HOMEWORK: 5.1, 6,12*,14*,16,26,32 (extension until Tuesday due to midterm on wednesday)

DEFINITIONS.

Linear space $0 \in X$, $\vec{x},\vec{y} \in X, \lambda \in \mathbf{R} \Rightarrow \vec{x}+\vec{y} \in X$, $\lambda\vec{x} \in X$.
Matrix A is a $m \times n$ matrix, it has n columns and m rows
Square matrix $n \times n$ matrix.
Vector $n \times 1$ matrix = column vector, $1 \times n$ matrix = row vector.
Linear transformation $T : \mathbf{R}^n \mapsto \mathbf{R}^m$, $\vec{x} \mapsto A\vec{x}$.
Column vectors of A are images of standard basis vectors $\vec{e}_1, \dots, \vec{e}_n$.
Linear system of equations $A\vec{x} = \vec{b}$, m equations, n unknowns.
Consistent system $A\vec{x} = \vec{b}$: "Good" case: for every $\vec{b}$ there is exactly one solution $\vec{x}$.
Vector form of linear equation $x_1\vec{v}_1 + \dots + x_n\vec{v}_n = \vec{b}$, $\vec{v}_i$ columns of A.
Matrix form of linear equation $\vec{w}_i \cdot \vec{x} = b_i$, $\vec{w}_i$ rows of A.
Augmented matrix of $A\vec{x} = \vec{b}$ is the matrix $[A|b]$ which has one column more as A.
Coefficient matrix of $A\vec{x} = \vec{b}$ is the matrix A.
Partitioned matrix matrix, where each entry is a matrix.
Matrix multiplication $[AB]_{ij} = \sum_n A_{in}B_{nj}$, dot product of i-th row in A with j'th column in B.
Gauss-Jordan elimination $A \to \text{rref}(A)$ in row reduced echelon form.
Gauss-Jordan elimination steps: swapping rows, scaling rows, adding rows to other rows.
Row reduced echelon form: every nonzero row has leading 1, columns with leading 1 have only 0 entries beside leading 1, every pivotal column to the left has leading 1 further to the left.
Pivot column column with leading 1 in $\text{rref}(A)$.
Rank of matrix A. Number of leading 1 in $\text{rref}(A)$. It is equal to $\dim(\text{im}(A))$.
Kernel of linear transformation T $\{\vec{x} \in \mathbf{R}^n, A\vec{x} = \vec{0}\}$.
Image of linear transformation T $\{A\vec{x}, \vec{x} \in \mathbf{R}^n\}$.
Inverse of T Linear transformation satisfying $ST = Id$. Corresponding matrix $B = A^{-1}$.
Rotation in plane $\vec{x} \mapsto A\vec{x}$, $A = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$, counterclockwise rotation by angle ϕ.
Dilation in Plane $\vec{x} \mapsto \lambda\vec{x}$, also called scaling.
Rotation-Dilation $\vec{x} \mapsto A\vec{x}$, $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.
Horizontal shear $\vec{x} \mapsto A\vec{x}$, $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$.
Vertical shear $\vec{x} \mapsto A\vec{x}$, $A = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$.
General Shear T leaves line L invariant and $A\vec{x} - \vec{x}$ parallel to L.
Reflection at line $\vec{x} \mapsto A\vec{x}$, $A = \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix}$.
Projection onto line containing unit vector $\vec{v}$ $\vec{x} \mapsto (\vec{x} \cdot \vec{v})\vec{v}$.
$\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ **span** X: Every $\vec{x} \in X$ can be written as $\vec{x} = \sum_i a_i\vec{v}_i$.
$\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ **linear independent** X: $\vec{0} = \sum_i a_i\vec{v}_i \Rightarrow a_1 = \dots = a_n = 0$.
$\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ **basis in** X: linear independent in X and span X.
Dimension of X Number of elements in a basis of X.
$\mathcal{B}$ **coordinates** $[\vec{v}]_{\mathcal{B}} = S^{-1}\vec{v}$, where $S = [\vec{v}_1, \dots, \vec{v}_n]$ contains basis vectors $\vec{v}_i$ as columns.
$\mathcal{B}$ **matrix of** T The matrix is $B = [[T(\vec{v}_1)]_{\mathcal{B}}, \dots, [T(\vec{v}_n)]_{\mathcal{B}}] = S^{-1}AS$.

RESULTS.

Linear transformations. T is linear if and only if $T(\vec{0}) = \vec{0}, T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y}), T(\lambda\vec{x}) = \lambda T(\vec{x})$.
Good-Bad-Ugly. A linear system of equations has either exactly 1, no or infinitely many solutions.
Rank-Nulletly theorem. $\dim(\ker(A)) + \dim(\text{im}(A)) = n$, where A is $m \times n$ matrix.
Behavior of kernel under G-J elimination. The kernel stays invariant under Gauss-Jordan elimination.
Behavior of image under G-J elimination. The image in general changes during Gauss-Jordan elimination.
Basis of image of A The Pivot columns of A form a basis of the image of A.
Basis of kernel of A Introduce a free variable for each non-Pivot column of A.
Inverse of 2×2 **matrix** Switch diagonal elements, change sign of wings and divide by determinant.
Kernel of compositions The kernel of A is contained in the kernel of BA.
Image of compositions The image of BA is contained in the image of B.
Matrix algebra $(AB)^{-1} = B^{-1}A^{-1}$, $A(B+C) = AB + AC$, etc. Note that $AB \neq BA$ in general.
A is invertible $\Leftrightarrow \text{rref}(A) = 1_n \Leftrightarrow$ columns form basis $\Leftrightarrow A\vec{x} = \vec{b}$ is consistent.

PROBLEM (Final Fall 2001, Topic Image and Kernel) You know $B = \text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Find (if possible) kernel, image and their dimensions.

SOLUTION. The linear system to B is $x + 2y + 5z = 0, z + 3u = 0$. Columns 2,4 are non pivotal. Solving gives $u = t, z = -3t, y = s, x = -2s - 5t$, so that a general basis element is $t[-5, 0, -3, 1] + s[-2, 1, 0, 0]$. We can not determine a basis for the image because we don't know A. The dimensions of the image and kernel are 2.

PROBLEM (First Midterm Fall 2001, Inverse) How many linear transformations in the plane have the property of being its own inverse?

SOLUTION. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. $A = A^{-1}$ means $A^2 = 1$ so that $\begin{bmatrix} a^2 + bc & b(a+d) \\ c(a+d) & bc + d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. This gives $a^2 = d^2$. The case $a = d = 0$ gives $bc = 1$. The case $a = d \neq 0$ gives $2ab = 0, 2ac = 0$ so that A is a multiple of the identity matrix 1_2. The case $a = -d \neq 0$ leaves us with $a^2 + bc = 1$. If we fix $a \in [-1, 1]$ the variables b, c have to satisfy $bc = 1 - a^2$.

Case 1) $a = d = \lambda$ $b = c = 0$	Case 2) $a = -d \in [-1, 1]$ $bc = 1 - a^2$

PROBLEM (First midterm Spring 2001, Linear transformations) Let A be a shear along a line L. Find $\ker(A - 1)$, $\text{im}(A - 1)$ and $(A - 1)^2$.

SOLUTION. Shears have the property that $(A-1)x = Ax - x$ is parallel to L, therefore $\text{im}(A - 1)$ is L. The kernel of $A - 1$ consists of vectors $Ax = x$. Because every $v \in L$ has this property, the kernel is L too. Both have dimension 1. We have $(A-1)^2 = 0$ because the image of $A - 1$ is a subset of the kernel.

PROBLEM (compare problem 47 section 3.4 page 145, or problem 5 in Spring 1999). Let L be the line spanned by $\vec{v} = (0.5, 0.5, 0)$ and let T be the counterclockwise rotation about an angle $\pi/2$ around L (this means $\pi/2$ clockwise if you look from $\vec{v}$ to the origin) (Find the matrix A.

SOLUTION. Draw a good picture. $\vec{e}_1$ goes to $[1/2, 1/2, 1/\sqrt{2}]$, $\vec{e}_2$ goes to $[1/2, 1/2, -1/\sqrt{2}]$, $\vec{e}_3$ goes to $[-1/\sqrt{2}, 1/\sqrt{2}, 0]$. These are the columns of A, so that $A = \begin{bmatrix} 1/2 & 1/2 & -1/\sqrt{2} \\ 1/2 & 1/2 & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}$.

PROBLEM (First midterm Spring 2000, problem 4) Let A be a 3×3 matrix satisfying $A^2 = 0$. Show that the image of A is a subset of the kernel of A and determine all possible values for $\text{rank}(A)$. Give an example.

SOLUTION. If x is in the image then $x = Ay$ and $Ax = AAy = 0$ so that x is in the kernel. A is not invertible and can not have rank 3. It can be the 0 matrix with rank 0. It can have rank 1: take 0 in the first two columns and e_1 in the last. It can not have rank 2 because the dimension of the kernel would be 1 and could not contain the kernel.

PROBLEM (first midterm Spring 2000, problem 6) Let $A = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$ Find B satisfying $B^2 = A$, determine $\text{rank}(B)$ and B^{17}.

SOLUTION. A is a rotation-dilation, a composition of a rotation by $\pi/2$ and dilation by 4. Take B as a rotation dilation with angle $\pi/4$ and dilation factor 2. The rank of B is 2 because if it were smaller, then also the rank of A were smaller. B^{17} is a rotation dilation with angle $17\pi/2 \sim \pi/2$ and dilation factor 2^{17}.

PROBLEM (problem 43 in section 3.3) If $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in $\mathbf{R}^n$ are linearly independent, are $\vec{w}_1 = \vec{v}_1, \vec{w}_2 = \vec{v}_1 + \vec{v}_2, \vec{w}_3 = \vec{v}_1 + \vec{v}_2 + \vec{v}_3$ also linearly independent?

SOLUTION. Yes: the linear map which maps the span of $\vec{v}_i$ into the span of $\vec{w}_i$ is invertible: it has the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

PROBLEM Problem 44 in 2.2: given n, find the rank of the matrix in which the entries are enumerated, column by column like in $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

SOLUTION. Deleting the first row from each other row makes the lower $n-1$ rows all linearly dependent. The rank is 2.