2/28/2002, COORDINATES **Math 21b, O. Knill**

HOMEWORK: 3.3: 22,24,32,36,40,56*, 3.4: 4,14,16,22,32*,48

SUMMARY LAST HOUR:

LINEAR SPACE. X **linear space**: $\vec{0} \in X$ closed under addition and scalar multiplication.
Examples: $X = \ker(A), X = \mathrm{im}(A)$ are linear spaces.

BASIS. $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\} \subset X$
$\mathcal{B}$ **linear independent**: $a_1\vec{v}_1 + ... + a_n\vec{v}_n = 0, \Rightarrow a_1 = ... = a_n = 0$.
$\mathcal{B}$ **span** X: $\vec{v} \in X$ then $\vec{v} = a_1\vec{v}_1 + ... + a_n\vec{v}_n$.
$\mathcal{B}$ **basis**: both linear independent and span.

DIMENSION. The number of elements in a basis in X is independent of the choice of the basis and called **dimension** of X.

UNIQUE REPRESENTATION. $\vec{v}_1, \dots, \vec{v}_n \in X$ **basis** $\Rightarrow$ every $\vec{v} \in X$ can be written uniquely as a sum $\vec{v} = a_1\vec{v}_1 + \dots + a_n\vec{v}_n$.

BASIS DEFINES INVERTIBLE MATRIX . $v_1, \dots, v_n$ form a basis in $\mathbf{R}^n \Leftrightarrow S = \begin{bmatrix} | & \dots & | \\ \vec{v}_1 & \dots & \vec{v}_n \\ | & \dots & | \end{bmatrix}$ is invertible.

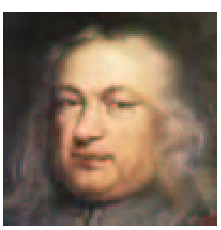
BASIS OF IMAGE A basis of the image of a matrix A is obtained by the pivotal columns of A.

BASIS OF KERNEL. rewrite rref(A) as a system of linear equations and introduce a free variable for every nonpivotal column.

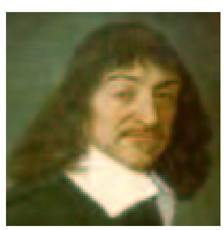
$\mathcal{B}$-COORDINATES. Given a basis $\vec{v}_1, \dots \vec{v}_n$, define the matrix $S = \begin{bmatrix} | & \dots & | \\ \vec{v}_1 & \dots & \vec{v}_n \\ | & \dots & | \end{bmatrix}$. It is invertible. If $\vec{x} = \sum_i c_i\vec{v}_i$, then c_i are called the **$\mathcal{B}$-coordinates** of $\vec{v}$. We write $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \dots \\ c_n \end{bmatrix}$. If $\vec{x} = \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix}$, we have $\vec{x} = S([\vec{x}]_{\mathcal{B}})$. In other words:

$\mathcal{B}$-coordinates of $\vec{x}$ are obtained by applying S^{-1} to the coordinates of the standard basis.

COORDINATES HISTORY. Cartesian geometry was introduced by Fermat and Descartes (1596-1650) around 1636. It had a large influence on mathematics. Algebraic methods were introduced into geometry. The beginning of the vector concept came only later at the beginning of the 19'th Century with the work of Bolzano (1781-1848). The full power of coordinates becomes possible if we allow to chose our coordinate system adapted to the situation.
Decartes biography shows how far dedication to the teaching of mathematics can go ...)
(...) In 1649 Queen Christina of Sweden persuaded Descartes to go to Stockholm. However the Queen wanted to draw tangents at 5 a.m. in the morning and Descartes broke the habit of his lifetime of getting up at 11 o'clock. After only a few months in the cold northern climate, walking to the palace at 5 o'clock every morning, he died of pneumonia.



Fermat

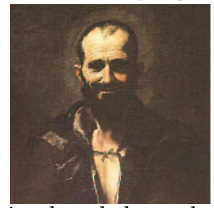


Decartes

Bolzano

EXAMPLE. Find the coordinates of $\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ with respect to the basis $\mathcal{B} = \{\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$.
We have $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $S^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. Therefore $[v]_{\mathcal{B}} = S^{-1}\vec{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Indeed $-1\vec{v}_1 + 3\vec{v}_2 = \vec{v}$.

CREATIVITY THROUGH LAZINESS? Decartes (1596-1650) introduced coordinates while lying on the bed, watching a fly (around 1630). Archimedes (285-212 BC) discovered a method to find the volume of bodies while relaxing in the bath. Newton (1643-1727) discovered Newton's law while lying under an apple tree.



According to Harvards David Perkins: "The Heureka effect", many creative breakthroughs have in common:

- Long search without apparent progress.
- Prevailing moment and break through.
- Transformation and realisation.

According to this pattern, having a breakthrough in a lazy moment is typical - but only after a long struggle and hard work.

$\mathcal{B}$-MATRIX. If $\mathcal{B} = \{v_1, \dots, v_n\}$ is a basis in $\mathbf{R}^n$ and T is a linear transformation on $\mathbf{R}^n$, then the $\mathcal{B}$-matrix of T is $B = \begin{bmatrix} | & \dots & | \\ [T(\vec{v}_1)]_{\mathcal{B}} & \dots & [T(\vec{v}_n)]_{\mathcal{B}} \\ | & \dots & | \end{bmatrix}$.

EXAMPLE. Let T be the reflection at the plane $x + 2y + 3z = 0$. Find the transformation matrix B in the basis $\vec{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$. Because $T(\vec{v}_1) = \vec{v}_1 = [\vec{e}_1]_{\mathcal{B}}, T(\vec{v}_2) = \vec{v}_2 = [\vec{e}_2]_{\mathcal{B}}, T(\vec{v}_3) = -\vec{v}_3 = -[\vec{e}_3]_{\mathcal{B}}$, the solution is $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

SIMILARITY. The $\mathcal{B}$ matrix of A is $B = S^{-1}AS$, where $S = \begin{bmatrix} | & \dots & | \\ \vec{v}_1 & \dots & \vec{v}_n \\ | & \dots & | \end{bmatrix}$. One says B is **similar** to A.

EXAMPLE. If A is similar to B, then A^2+A+1 is similar to $B^2+B+\mathbf{1}$. $B = S^{-1}AS, B^2 = S^{-1}B^2S, S^{-1}S = \mathbf{1}$, $S^{-1}(A^2 + A + 1)S = B^2 + B + \mathbf{1}$.

PROPERTIES OF SIMILARITY. A, B similar and B, C similar, then A, C are similar. If A is similar to B, then B is similar to A.

QUIZZ: If A is a 2×2 matrix and $S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then what is $S^{-1}AS$?

MORE ON THE CONJUGATION S. The transformation S^{-1} maps the coordinates from the standard basis into the coordinates of the new basis. In order to see what a transformation A does in the new coordinates, we map it back to the old coordinates, apply A and then map it back again to the new coordinates: $B = S^{-1}AS$.

The transformation in "bad" coordinates.

$$\begin{array}{ccc} \vec{v} & \stackrel{S}{\leftarrow} & \vec{w} \\ A \downarrow & & \downarrow B \\ A\vec{v} & \stackrel{S^{-1}}{\rightarrow} & B\vec{w} \end{array}$$

The transformation in "good" coordinates.

QUESTION. Can the matrix A belonging to a projection from $\mathbf{R}^3$ to a plane $x + y + 6z = 0$ be similar to a matrix which is a rotation by 20 degrees around the z axis? No: an invertible matrix can not be similar to a invertible matrix.

PROBLEM. Find a clever basis for the reflection of a light ray at the line $x+2y = 0$. $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.
SOLUTIon. Yo can achive $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. In the standard with $S = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$.

PROBLEM. All shears $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ with $a \neq 0$ are similar. SOLUTION. Use a basis $\vec{v}_1 = a\vec{e}_1$ and $\vec{v}_2 = \vec{e}_2$.