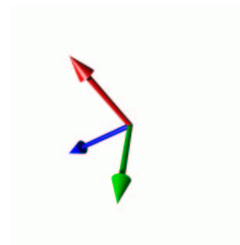
2/26/2002, BASIS — Math 21b, O. Knill

HOMEWORK: 3.2 6,18,28,36*,38*,48
(Preview: For next Tuesday: 3.3: 22,24,32,36,40,56*, 3.4: 4,14,16,22,32*,48)

DEFINITION BASIS. A set of vectors $\vec{v}_1, \dots, \vec{v}_m$ is a **basis** of a subspace X of $\mathbf{R}^n$ if they are **linear independent** and if they **span** the space X. Linear independent means that there are no nontrivial **linear relations** $a_i\vec{v}_1 + \dots + a_m\vec{v}_m = 0$. Spanning the space means that very vector $\vec{v}$ can be written as a linear combination $\vec{v} = a_1\vec{v}_1 + \dots + a_m\vec{v}_m$ of basis vectors. A **linear subspace** is a set containing $\{0\}$ which is closed under addition and scaling.



EXAMPLE. Three vectors in space which are not contained in a plane form a basis

FACT. If $\vec{v}_1, ..., \vec{v}_n$ is a basis, then every vector $\vec{v}$ can be represented **uniquely** as a linear combination of the $\vec{v}_i$.
$\vec{v} = a_1\vec{v}_1 + \dots + a_n\vec{v}_n$.

REASON. There is a representation because the vectors $\vec{v}_i$ span the space. If there were two different representations $\vec{v} = a_1\vec{v}_1 + \dots + a_n\vec{v}_n$ and $\vec{v} = b_1\vec{v}_1 + \dots + b_n\vec{v}_n$, then subtraction would lead to $0 = (a_1 - b_1)\vec{v}_1 + ... + (a_n - b_n)\vec{v}_n$. This nontrivial linear relation of the v_i is forbidden by assumption.

EXAMPLE. The vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ form a basis in the three dimensional space. If $\vec{v} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$, then $\vec{v} = \vec{v}_1 + 2\vec{v}_2 + 3\vec{v}_3$ and this representation is unique. We can find the coefficients by solving $A\vec{x} = \vec{v}$, where A has the v_i as column vectors. In our case, $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$ had the unique solution $x = 1, y = 2, z = 3$ leading to $\vec{v} = \vec{v}_1 + 2\vec{v}_2 + 3\vec{v}_3$.

FACT. If n vectors $\vec{v}_1, ..., \vec{v}_n$ span a space and $\vec{w}_1, ..., \vec{w}_m$ are linear independent, then $m \leq n$.

REASON. Assume $m > n$. Because $\vec{v}_i$ span, each vector $\vec{w}_i$ can be written as $\sum_j a_{ij}\vec{v}_j = \vec{w}_i$. After doing Gauss-Jordan elimination of the $m \times n$ matrix $\left| \begin{array}{ccc|c} a_{11} & \dots & a_{1n} & \vec{w}_1 \\ \dots & \dots & \dots & \dots \\ a_{m1} & \dots & a_{mn} & \vec{w}_m \end{array} \right|$ we end up with a matrix which has in the last line $\left| \begin{array}{ccc|c} 0 & ... & 0 & b_1\vec{w}_1 + ... + b_m\vec{w}_m \end{array} \right|$. The equation $b_1\vec{w}_1 + ... + b_m\vec{w}_m$ is a nontrivial relation between the $\vec{w}_i$. Contradiction.

DIMENSION. The number of elements in a basis is independent of the basis and called the **dimension**.

EXAMPLES. The dimension of $\{0\}$ is zero. The dimension of a line 1. The dimension of a plane is 2, the dimension of three dimensional space is 3. The dimension is independent on where the space is embedded in. For example: a line in the plane and a line in space have dimension 1.

A BASIS DEFINES AN INVERTIBLE MATRIX. The $n \times n$ matrix $A = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \\ | & | & & | \end{bmatrix}$ is invertible if and only if $\vec{v}_1, \dots, \vec{v}_n$ define a basis in $\mathbf{R}^n$.

EXAMPLE. In the example above, the 3×3 matrix A is invertible.

DIMENSION OF THE KERNEL. The number of column in rref(A) without leading 1's is the **dimension of the kernel** $\dim(\ker(A))$: we can introduce a parameter for each such column when solving $Ax = 0$ using Gauss-Jordan elimination.

DIMENSION OF THE IMAGE. The number of **leading 1** in rref(A), the rank of A is the **dimension of the image** $\dim(\text{im}(A))$ because every such leading 1 produces a different column vector (called **pivot column vectors**) and these column vectors are linearly independent.

DIMENSION FORMULA: ($A : \mathbf{R}^n \to \mathbf{R}^m$)

$$\boxed{\dim(\ker(A)) + \dim(\mathrm{im}(A)) = n}$$

EXAMPLE: A invertible $\Leftrightarrow$ the dimension of the image is n $\Leftrightarrow$ the dimension of the kernel 0.

PROOF. There are n columns. $\dim(\mathbf{ker}(A))$ is the number of columns without leading 1, $\dim(\mathrm{im}(A))$ is the number of columns with leading 1.

FINDING A BASIS FOR THE KERNEL. To solve $Ax = 0$, we bring the matrix A into the reduced row echelon form $\mathrm{rref}(A)$. For every non-leading entry in $\mathrm{rref}(A)$, we will get a free variable t_i. Writing the system $Ax = 0$ with these free variables gives us an equation $\vec{x} = \sum_i t_i \vec{v}_i$. The vectors $\vec{v}_i$ form a basis of the kernel of A.

REMARK. The problem to find a basis for all vectors $\vec{w}_i$ which are orthogonal to a given set of vectors, is equivalent to the problem to find a basis for the kernel of the matrix which has the vectors $\vec{w}_i$ in its rows.

FINDING A BASIS FOR THE IMAGE. Bring the $m \times n$ matrix A into the form $\mathrm{rref}(A)$. Call a column a **pivot column**, if it contains a leading 1. The corresponding set of column vectors of the original matrix A form a basis for the image because they are linearly independent and are in the image. Assume there are k of them. They span the image because there are $(k - n)$ non-leading entries in the matrix.

REMARK. The problem to find a basis of the subspace generated by $\vec{v}_1, \dots, \vec{v}_n$, is the problem to find a basis for the image of the matrix A with column vectors $\vec{v}_1, ..., \vec{v}_n$.

EXAMPLE. Two vectors on a line are linear dependent. The linear relation says that one is a multiple of the other. Three vectors in the plane are linear dependent. One can find a relation $a\vec{v}_1 + b\vec{v}_2 = \vec{v}_3$ by changing the size of the lengths of the vectors $\vec{v}_1, \vec{v}_2$ until $\vec{v}_3$ becomes the diagonal of the parallelogram spanned by $\vec{v}_1, \vec{v}_2$. Four vectors in three dimensional space are linearly dependent. As in the plane one can change the length of the vectors to make $\vec{v}_4$ a diagonal of the parallelepiped spanned by $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

EXAMPLE. Let A be the matrix $A = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix}$. In reduced row echelon form is $\mathrm{rref}(A) = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix}$. There is one **non-leading entry** in the second row. The dimension of the kernel is 1.

EXAMPLE. There are two column vectors with leading 1. The dimension of the image is 2. The dimension formula $2 + 1 = 3$ is satisfied.

EXAMPLE. $B = \mathrm{rref}(A) = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix}$. To determine a basis of the kernel we write $Bx = 0$ as a system of linear equations: $x + y = 0, z = 0$. The variable y is the free variable. With $y = t$, $x = -t$ is fixed. The linear system $\mathrm{rref}(A)x = 0$ is solved by $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. So, $\vec{v} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ is a basis of the kernel.

EXAMPLE. Because the first and third vectors in $\mathrm{rref}(A)$ are columns with leading 1's, the first and third columns $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ of A form a basis of the image of A.

WHY BASIS VECTORS? Would it not be easier just to look at the standard basis vectors $\vec{e}_1, \dots, \vec{e}_n$ only? The reason for more general basis vectors is that they allow a more flexible adaptation at the situation. A person in Paris prefers a different set of basis vectors than a person in Boston. We will also see that in many applications, problems can be solved easier with the right basis.

For example, to describe the reflection of a ray at a plane, it is preferable to use two vectors in the plane, and one orthogonal to the plane. When looking at a rotation, it is good to have one basis vector in the axis of rotation, the other two orthogonal to the axis. Choosing the right basis will be important when studying differential equations.