2/19/2002, LINEAR TRANSFORMATIONS (III) — Math 21b, O. Knill

HOMEWORK: 2.4: 4,14,26*,28,40,48

MATRIX PRODUCT. If B is a $m \times n$ matrix and A is a $n \times p$ matrix, then BA is a $m \times p$ matrix with entries $(BA)_{ij} = \sum_{k=1}^n B_{ik} A_{kj}$.

EXAMPLE. If B is a 3×4 matrix, and A is a 4×2 matrix then BA is a 3×2 matrix.

$$B = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 1 & 8 & 1 \\ 1 & 0 & 9 & 2 \end{bmatrix}, \; A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \; BA = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 1 & 8 & 1 \\ 1 & 0 & 9 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 15 & 13 \\ 14 & 11 \\ 10 & 5 \end{bmatrix}.$$

COMPOSING LINEAR TRANSFORMATIONS. If $S : \mathbf{R}^m \to \mathbf{R}^n, x \mapsto Ax$ and $T : \mathbf{R}^n \to \mathbf{R}^p, x \mapsto Bx$ are linear transformations, then their composition $T \circ S$ is a linear transformation from $\mathbf{R}^m$ to $\mathbf{R}^p$. The corresponding matrix is the matrix product BA.

EXAMPLE. Find the matrix which is a composition of a rotation around the x-axes by $\pi/2$ followed by a rotation around the y-axes by $-\pi/2$.
SOLUTION. The first transformation has the property that $e_1 \to e_1, e_2 \to -e_3, e_3 \to e_2$, the second $e_1 \to -e_2, e_2 \to e_1, e_3 \to e_3$. If A is the matrix belonging to the first transformation and B the second, then BA is the matrix to the composition.
$B = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$, $BA = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$. The composition maps $e_1 \to -e_2 \to e_3 \to e_1$ is a rotation around a long diagonal.

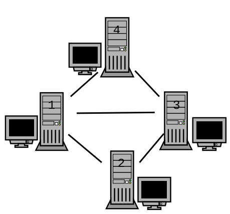
EXAMPLE. A rotation dilation is the composition of a rotation by $\alpha = \arctan(b/a)$ and a scale by $r = \sqrt{a^2 + b^2}$.

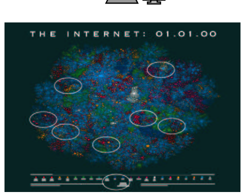
WHY? Matrix multiplication is a generalisation of usual multiplication of numbers or the dot product.

PROPERTIES. Note that $AB \neq BA$ in general! Otherwise, the same rules apply as for numbers: $A(BC) = (AB)C$, $AA^{-1} = A^{-1}A = 1_n$, $(AB)^{-1} = B^{-1}A^{-1}$, $A(B+C) = AB + AC$, $(B+C)A = BA + CA$ etc.

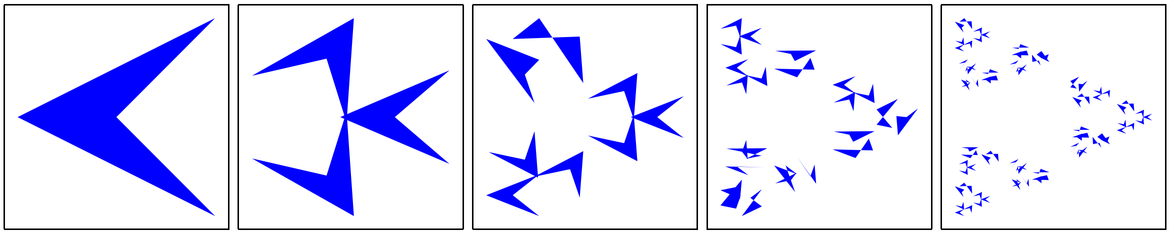
PARTITIONED MATRICES. The entries of matrices can themselves be matrices. If B is a $m \times n$ matrix and A is a $n \times p$ matrix, and assume the entries are $k \times k$ matrices, then BA is a $m \times p$ matrix where each entry $(BA)_{ij} = \sum_{k=1}^n B_{ik} A_{kj}$ is a $k \times k$ matrix. Partitioning matrices can improve matrix multiplication (i.e. Strassen algorithm).

EXAMPLE. If $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{00} & A_{22} \end{bmatrix}$, where A_{ij} are $k \times k$ matrices with the property that A_{11} and A_{22} are invertible, then $B = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1} A_{12} A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}$ is the inverse of A.



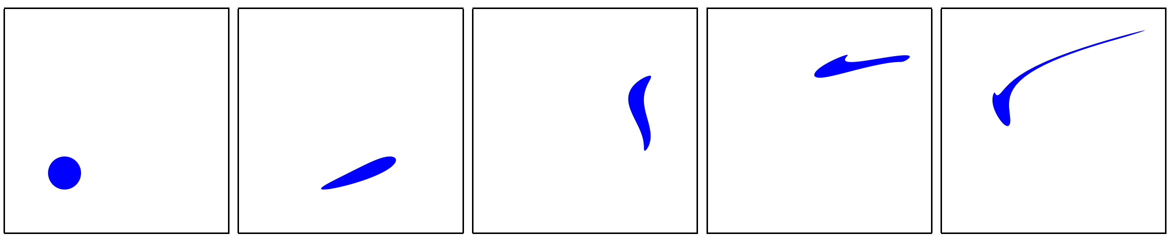


NETWORKS. Let us associate to the computer network (shown at the left) a matrix $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ To a worm in the first computer we associate a vector $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. The vector Ax has a 1 at the places, where the worm could be in the next step. The vector $(AA)(x)$ tells, in how many ways the worm can go from the first computer to other hosts in 2 steps. In our case, it can go in three different ways back to the computer itself.
Matrices help to solve combinatorial problems (see movie "Good will hunting"). For example, what does $[A^{1000}]_{22}$ tell about the worm infection of the network? What does it mean if A^{100} has no zero entries?



FRACTALS. Closely related to linear maps are **affine maps** $x \mapsto Ax + b$. They are compositions of a linear map with a translation. It is **not** a linear map if $B(0) \neq 0$. Affine maps can be disguised as linear maps in the following way: let $y = \begin{bmatrix} x \\ 1 \end{bmatrix}$ and the $(n+1)*(n+1)$ and $B = \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix}$, then $By = \begin{bmatrix} Ax+b \\ 1 \end{bmatrix}$.

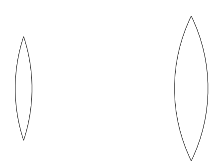
Fractals can be constructed by taking for example 3 affine maps f, g, h which contract area. For a given object Y_0 define $Y_1 = f(Y_0) \cup g(Y_0) \cup h(Y_0)$ and recursively $Y_k = f(Y_{k-1}) \cup g(Y_{k-1}) \cup h(Y_{k-1})$. Above you see Y_k after some iterations. In the limit, Y_k becomes a **fractal**, an object with noninteger dimension.



CHAOS. Consider a map in the plane like $T : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2x + 2\sin(x) - y \\ x \end{bmatrix}$ We apply this map again and again, we look at points $(x_1, y_1) = T(x, y)$, $(x_2, y_2) = T(T(x, y))$, etc. One writes T^n for the n-th iteration of the map and (x_n, y_n) for the image of (x, y) under the map T^n. The linear approximation of the map at a point (x, y) is the matrix $DT(x, y) = \begin{bmatrix} 2 + 2\cos(x) - 1 \\ 1 \end{bmatrix}$. (If $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$, then the row vectors of $DT(x, y)$ are just the gradients of f and g). T is called **chaotic at** (x, y), if the entries of $D(T^n)(x, y)$ grow exponentially fast with n. By the **chain rule**, $D(T^n)$ is the product of matrices $DT(x_i, y_i)$. For example, T is chaotic at $(0, 0)$. If there is a positive probability to hit a chaotic point, then T is called chaotic.

FALSE COLORS. Any color can be represented as a vector (r, g, b), where $r \in [0, 1]$ is the red $g \in [0, 1]$ is the green and $b \in [0, 1]$ is the blue component. Changing colors in a picture means applying a transformation on the cube. Let $T : (r, g, b) \mapsto (g, b, r)$ and $S : (r, g, b) \mapsto (r, g, 0)$. What is the composition of these two linear maps?

OPTICS. Matrices help to calculate the motion of light rays through lenses. A light ray $y(s) = x + ms$ in the plane is described by a vector (x, m). Following the light ray over a distance of lenght L corresponds to the map $(x, m) \mapsto (x + mL, m)$. In the lense, the ray is bent depending on the height x. The transformation in the lense is $(x, m) \mapsto (x, m - kx)$.



$$\begin{bmatrix} x \\ m \end{bmatrix} \mapsto A_L \begin{bmatrix} x \\ m \end{bmatrix} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ m \end{bmatrix}, \begin{bmatrix} x \\ m \end{bmatrix} \mapsto B_k \begin{bmatrix} x \\ m \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix} \begin{bmatrix} x \\ m \end{bmatrix}.$$

Examples:
1) Eye of length R looking far: $A_R B_k$. 2) Eye of length R looking at distance L: $A_R B_k A_L$. 3) Telescope: $B_{k_2} A_L B_{k_1}$.