HOMEWORK: 2.2: 4,8,10,32,47*,50*, 2.3: 10,20,26*,30,40,42*

INVERSE OF LINEAR TRANSFORMATION. If $A$ is a $n \times n$ matrix and $T : \vec{x} \mapsto Ax$ has an inverse $S$, then $S$ is linear and the $A^{-1}$, the matrix belonging to $S$ is called the **inverse** of $A$.

FINDING THE INVERSE. Let $1_n$ be the $n \times n$ matrix with 1 in the diagonal and zero elsewhere. Start with $[A|1_n]$ and make Gaussian elimination. Then

$$\text{rref}([A|1_n]) = [1_n|A^{-1}]$$

EXAMPLE. Find the inverse of $A = \begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix}$ with Gaussian elimination. Doing the Gauss-Jordan elimination to the right gives $A^{-1} = \begin{bmatrix} 2 & -3 \\ -1/2 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 2 & 6 & | & 1 & 0 \\ 1 & 4 & | & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & | & 1/2 & 0 \\ 1 & 4 & | & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & | & 1/2 & 0 \\ 0 & 1 & | & -1/2 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & | & 2 & -3 \\ 0 & 1 & | & -1/2 & 1 \end{bmatrix}$$

SHEAR:
$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ $\quad A^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

SCALING:
$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ $\quad A^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$

REFLECTION:
$A = \begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix}$ $\quad A^{-1} = \begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix}$

ROTATION:
$A = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(-\alpha) \end{bmatrix}$ $\quad A^{-1} = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$

ROTATION-DILATION:
$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ $\quad A^{-1} = \begin{bmatrix} a/r^2 & b/r^2 \\ -b/r^2 & a/r^2 \end{bmatrix}$, $r^2 = a^2 + b^2$

BOOST:
$A = \begin{bmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{bmatrix}$ $\quad A^{-1} = \begin{bmatrix} \cosh(\alpha) & -\sinh(\alpha) \\ -\sinh(\alpha) & \cosh(\alpha) \end{bmatrix}$

NONINVERTIBLE EXAMPLE. The projection $\vec{x} \mapsto A\vec{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is a non-invertible transformation.

MORE ON SHEARS. The shears $T(x, y) = (x + ay, y)$ or $T(x, y) = (x, y + ax)$ in $\mathbf{R}^2$ can be generalized. A shear is a linear transformation which fixes some line $L$ through the origin and which has the property that $T(\vec{x}) - \vec{x}$ is parallel to $L$ for all $\vec{x}$.

WHERE DO THEY APPEAR? Optics (see next week). Galileo transformation $(x, t) \mapsto (x + tv, t)$.

PROBLEM. $T(x, y) = (3x/2 + y/2, y/2 - x/2)$ is a shear along a line $L$. Find $L$.
SOLUTION. Solve the system $T(x, y) = (x, y)$. You find that the vector $(1, -1)$ is preserved.

MORE ON PROJECTIONS. A linear map $T$ with the property that $T(T(x)) = T(x)$ is a projection. Examples: $T(\vec{x}) = (\vec{y} \cdot \vec{x})\vec{y}$ is a projection onto a line spanned by a unit vector $\vec{y}$.

WHERE DO PROJECTIONS APPEAR? CAD: describe 3D objects using projections. A photo of an image is a projection. Compression algorithms like JPG or MPG or MP3 use projections (cut away the high frequencies).

MORE ON ROTATIONS. A linear map $T$ which preserves the angle between two vectors and the length of each vector is called a **rotation**. Rotations form an important class of transformations and will be treated later in more detail. In two dimensions, every rotation is of the form $x \mapsto A(x)$ with $A = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$.

EXAMPLES of rotations in three dimensions are $\vec{x} \mapsto Ax$, with $A = \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$, which is a rotation around the $z$ axis. or similar around any other major axes.

One can also get rotations by composing two reflections at planes. The axis of rotation is the intersection of the planes, the angle is twice the angle between the planes.

WHERE DO ROTATIONS APPEAR? Useful for example when putting an object into a form which is better manageable: Example: align a cylinder along the $z$-axis to make some computation. A rigid body centered at some point can only rotate.

MORE ON REFLECTIONS. Reflections are linear transformations different from the identity which are equal to their own inverse. Examples: in the plane **reflections at the origin**: $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, **reflections at a line** $A = \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix}$. In higher dimensions, the **reflection at a line** containing a unit vector $\vec{y}$ is $T(\vec{x}) = 2(\vec{x} \cdot \vec{y})\vec{y} - \vec{x}$.

Examples in 3D: reflections at the origin: $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. **reflections at a line** (for example the $z$ axis): $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. **reflections at a plane** (for example the $xy$-plane: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

WHERE DO REFLECTIONS APPEAR? Important symmetries in physics: T (time reflection), P (reflection at a mirror), C (change of charge) are reflections. It seems today that the composition of TCP is a fundamental symmetry in nature.

CHARACTERIZATION OF LINEAR TRANSFORMATIONS.

A linear transformation $T$ is linear if and only if $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ and $T(\lambda\vec{x}) = \lambda T(\vec{x})$ for all $\vec{x}, \vec{y}, \lambda$.

PROOF. If $\vec{x} = x_1\vec{e}_1 + ... + \vec{e}_n$, then $T(x) = x_1 T(\vec{e}_1) + ... + x_n T(\vec{e}_n) = x_1\vec{v}_1 + ... + x_n\vec{v}_n$. This can be rewritten as $T(\vec{x}) = A\vec{x}$, where $A$ is the matrix with vectors $v_j$ as columns.

EXAMPLE. Using the above criterium, show that the reflection $T(\vec{x}) = 2(\vec{x} \cdot \vec{y})\vec{y} - \vec{x}$ is a linear transformation.

EXAMPLE. Using the above criterium, show that the inverse of a linear transformation is linear.

DETERMINANT The determinant of a $2 \times 2$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad - bc$.

EXAMPLES. In the plane, rotations, boosts, shears have determinant 1, reflections have determinant -1.

THE INVERSE OF LINEAR MAPS $R^2 \mapsto R^2$.

There are explicit formulas for the inverse of invertible linear transformations (see later).

The formula in two dimensions can be remembered:

If $ad - bc \neq 0$, the inverse of a linear transformation $\vec{x} \mapsto Ax$ with $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $A^{-1} = \begin{bmatrix} d & -b \\ -d & a \end{bmatrix} /(ad - bc)$.
This can be checked directly.

PRO MEMORIAM: "To find the inverse, flip the diagonal, change sign of the wings and divide by the determinant."

COMPOSING LINEAR TRANSFORMATIONS. If $T : \mathbf{R}^n \to \mathbf{R}^m, x \mapsto Ax$ and $S : \mathbf{R}^m \to \mathbf{R}^k, x \mapsto Bx$ are linear transformations, then their composition $S \circ T$ is a linear transformation. (Later: The matrix of the composition is the matrix product $BA$).

EXAMPLE. A rotation dilation is a composition of a rotation by an angle $\alpha = \arctan(b/a)$ and a scale by factor $r = \sqrt{a^2 + b^2}$.