

This is **not** a list of topics covered in the course. It is just a selection of subjects, where linear algebra is useful. The aim is to convince you that it is worth learning this subject.

Graphs, Networks	<p>Linear algebra is used to understand networks. The adjacency matrix of a graph is defined by $A(i,j) = 1$ if there is an edge from vertex i to vertex j in a graph. Otherwise the entry is zero. A problem using such matrices appeared on a blackboard at MIT in the movie "Good will hunting".</p>	<p>Application: A_{ij}^n is the number of n-step walks in the graph which start at the vertex i and end at the vertex j.</p>
Chemistry, Mechanics	<p>Complicated objects like a bridge (example the one built over the Charles river at the "big dig"), or a molecule (i.e. a protein) can be modeled through finitely many parts (bridge elements or atoms) coupled with attractive and repelling forces. The vibrations of the system are then described by a differential equation $\dot{x} = Ax$, where x is a vector. Differential equations form an important part of this course.</p>	<p>The solution $x(t) = \exp(At)$ of the differential equation $\dot{x} = Ax$ can be understood and computed by finding the eigenvalues of the matrix A. Knowing these frequencies is important for the design of a mechanical object because the engineer can damp the dangerous frequencies. In chemistry or medicine, the knowledge of the vibration resonances allows to determine the shape of a molecule.</p>
Quantum computing	<p>A quantum computer is a quantum mechanical system used to perform computations. The state x of a machine is no more a sequence of bits like in a classical computer but a sequence of qubits, where each qubit is a vector. The memory can be represented as a vector. The computation is a multiplication $x \mapsto Ax$ with a suitable matrix A.</p>	<p>The memory of a quantum computer is a vector, the computation a linear operation on this vector. Theoretically, quantum computations could speed up conventional computations significantly. They could be used for example for cryptological purposes. Freely available QCL interpreters can simulate quantum computers with an arbitrary number of qubits.</p>
Chaos theory	<p>Dynamical system theory deals with the iteration of maps or solutions of differential equations. At each time t, one has a map $T(t)$ on the vector space. The linearisation $DT(t)$ is called Jacobean is a matrix. If the largest eigenvalue of $DT(t)$ grows exponentially in t, then the system shows "sensitive dependence on initial conditions" = "chaos".</p>	<p>Examples of dynamical systems are our solar system, stars in a galaxy, electrons in a plasma or particles in a fluid. The theoretical study is intrinsically linked to linear algebra because stability properties and the prediction of the system often depends on linearisation.</p>

Coding theory	<p>Coding theory is used for encryption or error correction. In the first case, the data x are mapped by a map T into code $y=Tx$. For a good code, T is a "trapdoor function" in the sense that it is hard to get x back, when y is known. In the second case, a code is a linear subspace of a vector space and T is the transmission with errors. The projection of Tx onto the subspace corrects the error.</p>	<p>Linear algebra enters in different ways, often directly because the objects are vectors but also indirectly like for example in algorithms which aim at cracking encryption schemes.</p>
Data compression	<p>Image- (i.e. JPG), video- (MPG4) and sound compression algorithms (i.e. MP3) make use of the Fourier transform which is special a linear transformation. In all cases, the compression makes use of the fact that in the Fourier space information can be cut away without disturbing the main information.</p>	<p>Typically, a picture, a sound or movie is cut into smaller junks. These parts are represented by vectors. If U denotes the Fourier transform and P is a cutoff function, then $y = PUx$ is transferred or stored on a CD or DVD. The receiver obtains back $U^T y$ which is close to x in the sense that the human eye or human ear does not notice a big difference.</p>
Optimization	<p>When extremizing a function f on data which satisfy a constraint $g(x) = 0$, the method of Lagrange multipliers asks to solve a linear system of equations $\nabla f(x) = \lambda \nabla g(x), g(x) = 0$ for the $(n + 1)$ unknowns (x, λ), where ∇f is the gradient of f.</p>	<p>While for a small number of variables, any method for solving the system could be taken, one has to use more sophisticated methods if the system has many variables.</p>
Computer games	<p>Moving around in a computer game like QUAKE requires rotations and translations to be implemented efficiently. Hardware acceleration can help to handle this.</p>	<p>Rotations are given by orthogonal matrices. For example if an object located at $(0,0,0)$, turning around the y-axes by an angle ϕ, every point in the object gets transformed by the matrix</p> $\begin{pmatrix} \cos(\phi) & 0 & \sin(\phi) \\ 0 & 1 & 0 \\ -\sin(\phi) & 0 & \cos(\phi) \end{pmatrix}$

Statistics	<p>When analyzing data statistically, one is often interested in the correlation matrix $A_{ij} = E[Y_i Y_j]$ of a random vector $X = (X_1, \dots, X_n)$ with $Y_i = X_i - E[X_i]$. This matrix is derived from the data and determines often the random variables when the type of the distribution is fixed.</p>	<p>For example, if the random variables have a Gaussian (=Bell shaped) distribution, the correlation matrix together with $E[X_i]$ determines the random variables.</p>
Game theory	<p>Games are often represented by pay-off matrices. These matrices tell the outcome, when the decision of each player is given.</p>	<p>A famous example is the prisoner dilemma. Each player has two choices, either to cooperate or not to cooperate. The game is described by a 2x2 matrix like for example $\begin{pmatrix} 3 & 0 \\ 5 & 1 \end{pmatrix}$. If a player cooperates and his partner also, both get 3 points. If his partner cheats and he cooperates, he gets 5 points. If both cheat, both get 1 point.</p>
Neural Nets	<p>In part of neural network theory, for example Hopfield networks, the state space is a 2n-dimensional vector space. Every state of the network is given by a vector x where each component takes values -1 or 1. If W is a symmetric nxn matrix, one has a "learning map" $T : x \mapsto \text{sign} Wx$, where sign is taken component wise. The energy of the state is the dot product $-(x, Wx)/2$. One is interested in fixed points of the map.</p>	<p>For example, if $W_{ij} = x_i y_j$, then x is a fixed point of the learning map.</p>

Markov Processes	<p>Suppose we have three bags with 10 balls each. Every time we throw a dice and a 5 shows up, we move a ball from bag 1 to bag 2, if the dice shows 1 or 2, we move a ball from bag 2 to bag 3, if 3 or 4 turns up, we move a ball from bag 3 to bag 1 and a ball from bag 3 to bag 2. What distribution of balls will we see in average?</p>	<p>This is a Markov chain described by a matrix $\begin{pmatrix} 5/6 & 1/6 & 0 \\ 0 & 2/3 & 1/3 \\ 1/6 & 1/6 & 2/3 \end{pmatrix}$. From this matrix, the equilibrium distribution can be read off as an eigenvector of a matrix.</p>
Splines	<p>In computer graphics used for example to construct cars, one wants to interpolate points with smooth curves. One example: assume you want to find a curve connecting two points P and Q and the direction is given at each point. Find a cubic function $f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$ which interpolates.</p>	<p>If we write down the conditions, we will have to solve a system of 4 equations for four unknowns. Graphic artists (i.e. at Pixar) need to have linear algebra skills!</p>
Symbolic dynamics	<p>Assume that a system can be in three different states a, b, c and transitions $a \mapsto b, b \mapsto a, b \mapsto c, c \mapsto c, c \mapsto a$ are allowed. A possible evolution of the system is then $a, b, a, b, a, c, c, c, a, b, c, a, \dots$. One calls this a description of the system with symbolic dynamics. This language is used in information theory, or dynamical systems theory.</p>	<p>The dynamics of the system is so coded with symbolic dynamics. One can describe the transition with a matrix $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$. Information theoretical quantities like the "entropy" can be read off from this matrix.</p>

The list of applications could be continued. Many inverse problems like tomography or problems in astrophysics or economics depend on linear algebra. Control theory, partial differential equations, general relativity, combinatorics are other fields.