What is Linear Algebra used for (I)? 2/4/2001 **Math 21b, Oliver Knill**

This is **not** a list of topics covered in the course. It is just a selection of subjects, where linear algebra is useful. The aim is to convince you that it is worth learning the subject.

Graphs, Networks	Linear algebra helps to understand networks. The **adjacency matrix** of a graph is defined by A(i,j) = 1 if there is an edge from vertex i to vertex j. Otherwise the entry is zero.	Application: A^n_{ij} is the number of n-step walks in the graph which start at the vertex i and end at the vertex j.
Chemistry, Mechanics	**Complicated objects** like a bridge (think of the one built over the Charles river at the "big dig"), or a molecule (i.e. a protein) can be modeled through finitely many parts (bridge elements or atoms) coupled with springs. The vibrations of the system are then described by a differential equation d/dt x = A x, where x is a possibly huge vector. Differential equations will be part of this course.	The solution $x(t) = exp(At)$ of the differential equation can be understood and computed by finding the eigenvalues of the matrix A. Knowing these frequencies is important for the design of a mechanical object because the engineer can damp the dangerous frequencies. In Chemistry or Medicine, the vibration spectrum (resonance) allows to determine the shape of a molecule.
Quantum computing	A **quantum computer** is a quantum mechanical system used to perform computations. The state x of a machine is no more a sequence of bits like in a classical computer but a sequence of qubits, where each qubit is a vector. The memory can be represented as a vector. The computation is a multiplication $x \mapsto Ax$ with a suitable matrix A. vspace0.5cm	The memory of a quantum computer is a vector, the computation a linear operation on this vector. Theoretically, quantum computations could speed up conventional computations significantly (for example for cryptological purposes). Freely available QCL interpreters can simulate quantum computers with an arbitrary number of qubits.
Chaos theory	**Dynamical system theory** deals with the iteration of maps or solutions of differential equations. At time t, one has a map $T(t)$ on the vector space. The linearisation $DT(t)$ is called Jacobean is a matrix. If the largest eigenvalue of $DT(t)$ grows exponentially in t, then the system shows "sensitive dependence on initial conditions" = "chaos".	Examples of dynamical systems are our solar system, stars in a galaxy, electrons in a plasma or particles in a fluid. The theoretical study is intrinsically linked to linear algebra because stability properties and the prediction of the system often depends on linearisation.

Coding theory	**Coding theory** is used for encryption or error correction. In the first case, the data x are maped by a map T into **code** y=Tx. For a good code, T is a "trapdoor function" in the sense that it is hard to get x back, when y is known. In the second case, a code is a linear subspace of a vector space and T is the transmission with errors. The projection of Tx onto the subspace corrects the error.	Linear algebra enters in different ways, often directly because the objects are vectors but also indirectly like for example in algorithms which aim at cracking encryption schemes.
Data compression	Image- (i.e. JPG), video- (MPG4) and sound **compression** algorithms (i.e. MP3) make use of the Fourier transform, a linear transformation. In all cases, the compression makes use of the fact that in the Fourier space information can be cut away without disturbing the main information.	Typically, a picture, sound or movie is cut into smaller junks. These parts are then represented by vectors. If U denotes the Fourier transform and P is a cut-off function, then $y = PUx$ is transferred or stored on a CD. The receiver obtains back $U^T y$ which is close to x in the sense that the human eye or or human ear does not notice a big difference.
Optimization	When **extremizing** a function f on data which satisfy a constraint $g(x) = 0$, the method of Lagrange multipliers asks to solve a linear system of equations $\nabla f(x) = \lambda \nabla g(x), g(x) = 0$ for the $(n+1)$ unknowns (x, l), where ∇f is the gradient of f.	While for a small number of variables, any method for solving the system could be taken, one has to use more sophisticated methods with many variables.
Computer games	Moving around in a **computer game** like QUAKE requires rotations and translations to be implemented fast. Hardware acceleration can help to handle this.	Rotations are given by orthogonal matrices. For example if an object located at $(0,0,0)$, turning around the y-axes by an angle ϕ, every point in the object gets transformed by the matrix $\begin{pmatrix} \cos(\phi) & 0 & \sin(\phi) \\ 0 & 1 & 0 \\ -\sin(\phi) & 0 & \cos(\phi) \end{pmatrix}$

Statistics	When analyzing data statistically, one is often interested in the **correlation matrix** $A_{ij} = E[Y_i Y_j]$ of a random vector $X = (X_1, \dots, X_n)$ with $Y_i = X_i - E[X_i]$. This matrix is derived from the data and determines often the random variables when the type of the distribution is fixed.	For example, if the random variables have a Gaussian (=Bell shaped) distribution, the correlation matrix together with $E[X_i]$ determines the random variables.
Game theory	**Games** are often represented by pay-off matrices. These matrices tell the outcome, when the decision of each player is given.	A famous example is the prisoner dilemma. Each player has two choices, either to corporate or not to corporate. The game is described by a 2x2 matrix like for example $\begin{pmatrix} 3 & 0 \\ 5 & 1 \end{pmatrix}$. If a player cooperates and his partner also, both get 3 points. If his partner cheats and he cooperates, he gets 5 points. If both cheat, both get 1 point.
Neural Nets	In part of **neural network** theory, for example Hopfield networks, the state space is a 2n-dimensional vector space. Every state of the network is given by a vector x where each component takes values -1 or 1. If W is a symmetric nxn matrix, one has a "learning map" $T : x \mapsto \text{sign}Wx$, where sign is taken component wise. The energy of the state is the dot product $-(x, Wx)/2$. One is interested in fixed points of the map.	For example, if $W_{ij} = x_i y_j$, then x is a fixed point of the learning map.

Markov Processes	Suppose we have three bags with 10 balls each. Every time we throw a dice and a 5 shows up, we move a ball from bag 1 to bag 2, if the dice shows 1 or 2, we move a ball from bag 2 to bag 3, if 3 or 4 turns up, we move a ball from bag 3 to bag 1 and a ball from bag 3 to bag 2. What distribution of balls will we see in average?	This is a Markov chain described by a matrix $\begin{pmatrix} 5/6 & 1/6 & 0 \\ 0 & 2/3 & 1/3 \\ 1/6 & 1/6 & 2/3 \end{pmatrix}$. From this matrix, the equilibrium distribution can be read off as an eigenvector of a matrix.
Splines	In **computer graphics** used for example to construct cars, one wants to interpolate points with smooth curves. One example: assume you want to find a curve connecting two points P and Q and the direction is given at each point. Find a cubic function $f(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$ which interpolates.	If we write down the conditions, we will have to solve a system of 4 equations for four unknowns. Graphic artists (i.e. at Pixar) need to have linear algebra skills!
Symbolic dynamics	Assume that a system can be in three different states a, b, c and transitions $a \mapsto b$, $b \mapsto a$, $b \mapsto c$, $c \mapsto c$, $c \mapsto a$ are allowed. A possible evolution of the system is then $a, b, a, b, a, c, c, c, a, b, c, a...$ One calls this a description of the system with **symbolic dynamics**. This language is used in information theory, or dynamical systems theory.	The dynamics of the system is so coded with symbolic dynamics. One can describe the transition with a matrix $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$. Information theoretical quantities like the "entropy" can be read off from this matrix.

The list of applications could be continued. Many inverse problems like tomography or problems in astrophysics or economics depend on linear algebra. Control theory, partial differential equations, general relativity, combinatorics are other fields.