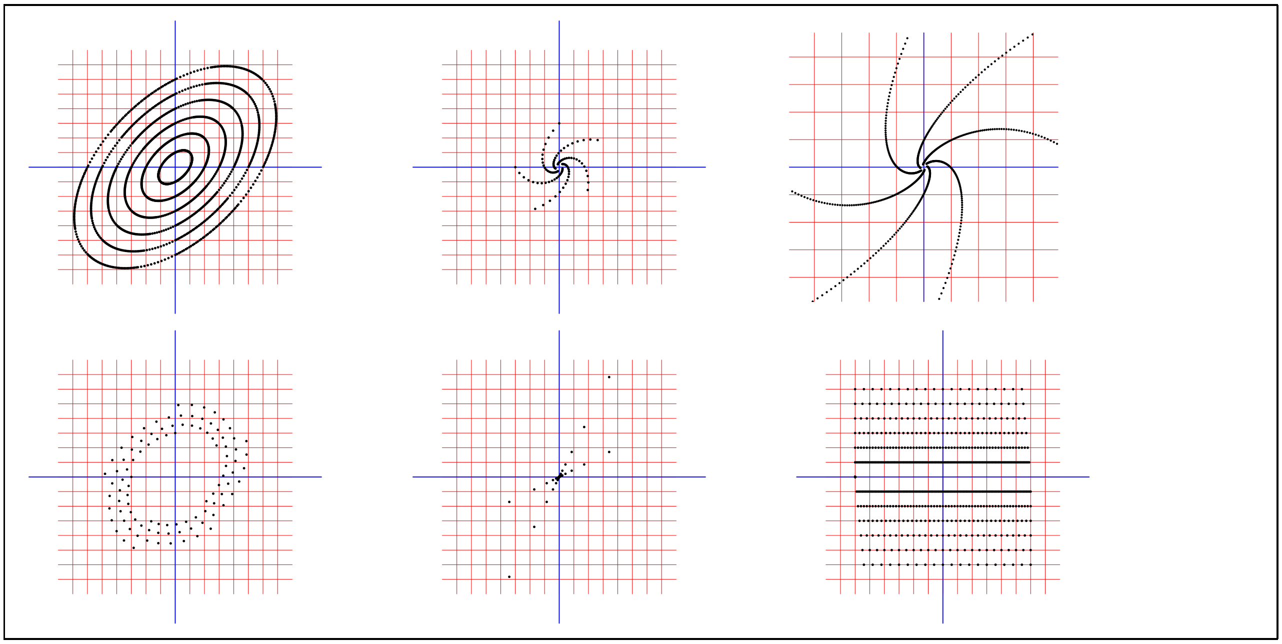
Homework for Monday, April 9, 2001: Section 6.5, 8,12,22,28,40afg,42*

LINEAR DYNAMICAL SYSTEM. A linear map $x \mapsto Ax$ defines a **dynamical system**. Iterating the map produces an **orbit** $x_0, x_1 = Ax, x_2 = A^2 = AAx, ...$ The vector $x_n = A^n x_0$ describes the situation of the system at **time** n.
QUESTIONS. We are interested in questions like:
Where does x_n go when time evolves? Can one describe what happens asymptotically when time n goes to infinity?
In the case of the Fibonnacci sequence x_n which gives the number of rabbits in a rabbit population in the year n, the population grows essentially exponentially. Linear algebra (see the homework) gave us the rate as the golden mean: $x_{n+1}/x_n = (\sqrt{5}+1)/2$. Such a behavior would be called **unstable**.



ASYMPTOTIC STABILITY. The origin 0 is left invariant under a linear map $A : x \mapsto Ax$. It is called **asymptotically stable** if $A^n(x) \to 0$ for all $x \in \mathbb{R}^n$.

EXAMPLE 1. Remember the chemical spill into the Swiss lakes? There all $A^n(x) \to 0$, where x is a three dimensional vector giving the amount of chemicals in each of the three lakes.

EXAMPLE 2. Let $A = \begin{bmatrix} p & -q \\ q & p \end{bmatrix}$ be a dilation rotation matrix. We have seen in class last time that multiplication wich such a matrix is analogue to the multiplication of complex numbers $p+iq$. So, A^n is the dilation rotation matrix corresponding to $(p+iq)^n$. Now, $|(p+iq)|^n = |p+iq|^n$ and the origin is asymptotically stable if and only if $|p+iq| < 1$. Because $\det(A) = |p+iq|^2$, rotation-dilation matrices A have an asymptotical stable origin if and only if $|\det(A)| < 1$.

EXAMPLE 3. If a matrix A has an eigenvalue $|\lambda| \geq 1$ to the eigenvector v, then $A^n v = \lambda^n v$ whose length is $|\lambda^n|$ times the length of v. So, we have no asymptotic stability if an eigenvalue satisfies $|\lambda| \geq 1$.

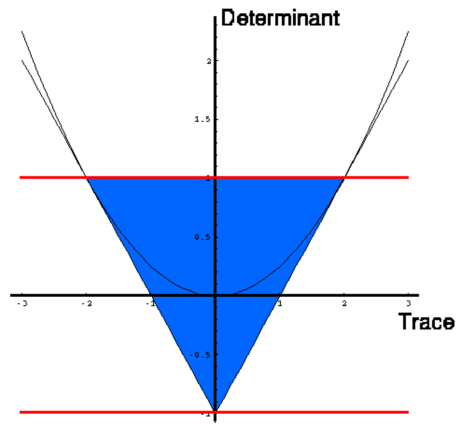
STABILITY. The book also writes "stable" for "asymptotically stable". This is ok to abbreviate. Note however that commonly the term "stable" also includes rotations or the identity. We would urge to leave always the attribute "asymptotic".

DILATION-ROTATION MATRICES. Dilation-rotation matrices $\begin{bmatrix} p & -q \\ q & p \end{bmatrix}$ have the eigenvalues $p \pm iq$.

ROTATIONS. Rotations $\begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$ have the eigenvalue $\exp(\pm i\phi) = \cos(\phi) + i\sin(\phi)$.
DILATIONS. Dilations $\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$ have the eigenvalue r with algebraic and geometric multiplicity 2.

CRITERION. A linear dynamical system $x \mapsto Ax$ has an asymptotically stable origin if and only if all eigenvalues have norm < 1.
PROOF. We have already seen in Example 3, that if one eigenvalue satisfies $|\lambda| > 1$, then the origin is not asymptotically stable. In the case when all eigenvalues are different, we have an eigenbasis $v_1, \dots, v_n$. Every x can be written as $x = \sum_{j=1}^n x_j v_j$. Then, $A^n x = A^n(\sum_{j=1}^n x_j v_j) = \sum_{j=1}^n x_j \lambda_j^n v_j$ and because $|\lambda_j|^n \to 0$, there is stability. The proof of the general case will be accessible later.

THE 2-DIMENSIONAL CASE. The characteristic polynomial of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $f_A(\lambda) = \lambda^2 - \mathrm{tr}(A)\lambda + \det(A)$. The eigenvalues are $\lambda_\pm = \mathrm{tr}(A)/2 \pm \sqrt{(\mathrm{tr}(A)/2)^2 - \det(A)}$, the eigenvectors are $v_\pm = \begin{bmatrix} \lambda_\pm - d, c \end{bmatrix}$. If the discriminant $(\mathrm{tr}(A)/2)^2 - \det(A)$ is nonnegative, then the eigenvalues are real. This happens below the parabola, where the discriminant is zero.



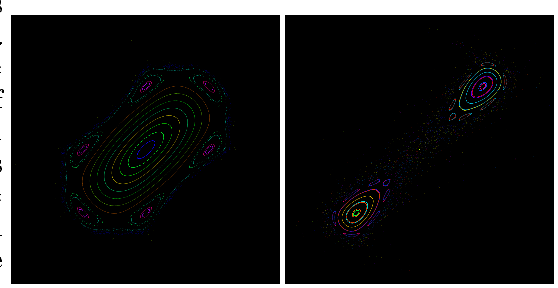
CRITERION. A linear dynamical system in two dimensions is asymptotically stable if and only if $(\mathrm{tr}(A), \det(A))$ is inside the **stability triangle** bounded by the lines $\det(A) = 1, \det(A) = \mathrm{tr}(A) - 1$ and $\det(A) = -\mathrm{tr}(A) - 1$.
PROOF. Write $T = \mathrm{tr}(A)/2, D = \det(A)$. If $|D| \geq 1$, there is no asymptotic stability. If $\lambda = T + \sqrt{T^2 - D} = \pm 1$, then $T^2 - D = (\pm 1 - T)^2$ and $D = 1 \pm 2T$. For $D \leq -1 + |2T|$ we have a real eigenvalue ≥ 1. The conditions $D > |2T| - 1$ include $D > -1$ so that the triangle can be described shortly as $|\mathrm{tr}(A)| - 1 < \det(A) < 1$.

EXAMPLES.
1) The matrix $A = \begin{bmatrix} 1 & 1/2 \\ -1/2 & 1 \end{bmatrix}$ has the determinant $5/4$ and trace 2. It has an unstable origin. It is a dilation-rotation matrix which corresponds to the complex number $1 + i/2$ which has an absolute value > 1.
2) A rotation is never asymptotically stable: The determinant is 1 and the trace is $2\cos(\phi)$. Rotations are the upper side of the **stability triangle**.
3) A dilation is asympotically stable if and only if the scaling factor has norm < 1.

SOME PROBLEMS.
1) If A is a matrix with has an asymptotically stable origin, what is the stability of the origin with respect to the system A^T?
2) If A is a matrix which has an asymptotically stable origin, what is the stability with respect to to A^{-1}?
3) If A is a matrix which has an asymptotically stable origin, what is the stability with respect to to A^{100}?

ON THE STABILITY QUESTION.
For general dynamical systems, the question of stability can be very difficult. In this course, we only deal with linear dynamical systems, where the eigenvalues determine the whole picture. For nonlinear systems, the story is not so simple even for simple maps like the Henon map (see below). The questions go deeper: it is for example not known, whether our solar system is stable. We don't know whether in some future, one of the planets can get expelled from the solar system (this is of course a mathematical question because the time it would need for that would be larger than the life time of the sun).
For other dynamical systems like the atmosphere of the earth or the stock market, we would really like to know what happens in the near future ...

A pioneer in stability theory was Aleksandr Lyapunov (1857-1918). For nonlinear systems like $x_{n+1} = gx_n - x_n^3 - x_{n-1}$ the stability of the origin is nontrivial. As with Fibonacci, this can be written as $(x_{n+1}, x_n) = (gx_n - x_n^2 - x_{n-1}, x_n) = A(x_n, x_{n-1})$ called **cubic Henon map** in the plane. To the right are orbits in the cases $g = 1.5$, $g = 2.5$.



The first case is stable (but proving this requires a theory called KAM), the second case is unstable (in this case actually the linearization at 0 determines the picture).