

Homework for Monday, March 11, 2001: Section 4.3, 2,6,8,18*,20,34cdefg*

DEFINITION The **transpose** of a matrix A is the matrix $(A^T)_{ij} = A_{ji}$. If A is a $n \times m$ matrix, then A^T is a $m \times n$ matrix. For square matrices, the transposed matrix is obtained by reflecting the matrix at the diagonal.

EXAMPLES The transpose of a vector $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is the row vector $A^T = [1 \ 2 \ 3]$.
 The transpose of the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is the matrix $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$.

A PROPERTY OF THE TRANSPOSE.

- a) If x, y are two vectors, then $x \cdot Ay = A^T x \cdot y$. b) $(AB)^T = B^T A^T$. c) $(A^T)^T = A$.
- a) Because $x \cdot Ay = \sum_j \sum_i x_i A_{ij} y_j$ and $A^T x \cdot y = \sum_j \sum_i A_{ji} x_i y_j$ the two expressions are the same by renaming i and j .
- b) $(AB)_{kl} = \sum_i A_{ki} B_{il}$. $(AB)^T_{kl} = \sum_i A_{li} B_{ik} = A^T B^T$.
- c) $((A^T)^T)_{ij} = (A^T)_{ji} = A_{ij}$.

DEFINITION. A $n \times n$ matrix A is called **orthogonal** if $A^T A = 1$. The corresponding linear transformation is called **orthogonal**.

EXAMPLES. The rotation matrix $A = \begin{bmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{bmatrix}$ is orthogonal because

$$A^T A = \begin{bmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{bmatrix} \cdot \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

A reflection at a line is an orthogonal transformation because $A^T A = \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix}$.
 $\begin{bmatrix} \cos(2\phi) & -\sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

FACTS. An orthogonal transformation preserves the dot product: $Ax \cdot Ay = x \cdot y$. Proof: we use the above property of the transpose, we see that the left hand side is $A^T Ax \cdot y$ and because of the orthogonality property, this is $x \cdot y$.

Consequently, an orthogonal transformation preserves the **length** of vectors as well as the **angles** between them. The reason is that We have $\|Ax\|^2 = Ax \cdot Ax = x \cdot x \|x\|^2$. Let α be the angle between x and y and β denote the angle between Ax and Ay . Using a) and the property of the dot product, we get $\|Ax\| \|Ay\| \cos(\beta) = Ax \cdot Ay = x \cdot y = \|x\| \|y\| \cos(\alpha)$. Because of b), this means $\cos(\alpha) = \cos(\beta)$.

ORTHOGONAL MATRICES AND BASIS. A linear transformation A is orthogonal if and only if the column vectors of A form an orthonormal basis. (That is what $A^T A = 1$ means.)

COMPOSITION OF ORTHOGONAL TRANSFORMATIONS. The composition of two orthogonal transformations is orthogonal. The inverse of an orthogonal transformation is orthogonal. Proof. The properties of the transpose give $(AB)^T AB = B^T A^T AB = B^T B = 1$ and $(A^{-1})^T A^{-1} = (A^T)^{-1} A^{-1} = (AA^T)^{-1} = 1$.

EXAMPLES. The composition of two reflections at a line is a rotation. The composition of two rotations is a rotation.

ORTHOGONAL PROJECTIONS. The orthogonal projection P onto a linear space with basis v_1, \dots, v_n is the matrix AA^T , where A is the orthogonal matrix with column vectors v_i . To see this just translate the formula $Px = (w_1 \cdot x)w_1 + \dots + (w_n \cdot x)w_n$ into the language of matrices: $A^T x$ is a vector with components $b_i = (w_i \cdot x)$ and Ab is the sum of the $b_i w_i$, where w_i are the column vectors of A .

WHY DO WE CARE ABOUT ORTHOGONAL TRANSFORMATIONS?

- Galileo transformations in physics are compositions of translations with orthogonal transformations.
- Many coordinate transformations are orthogonal transformations.
- In the QR decomposition of a matrix A , the matrix Q is orthogonal. Because $Q^{-1} = Q^t$, this allows to invert A easier.
- Many transformations which have applications (i.e. Fourier transformation) are orthogonal transformations.
- Quantum mechanical evolutions (when written as real matrices) are orthogonal transformations.

WHICH OF THE FOLLOWING MAPS ARE ORTHOGONAL?:

- Yes No Shear
- Yes No Projection in three dimensions onto a plane.
- Yes No Reflection in two dimensions at the origin.
- Yes No Reflection in three dimensions at a plane.
- Yes No Dilation with factor 2.
- Yes No The Lorentz boost in the plane.
- Yes No A translation.