Lesson 8 ker(A) and im(A): Some problems 2/23/2001, Math21b, O.Knill

Homework for Monday: 10,22,34,38*,44,48*

G. Polya (1887-1985 see photo) wrote a nice book, "how to solve it". In it, there is some general advise on "how to solve problems". Let us look at some problems in the context of kernel and image with this advise in mind: Polya's scheme is sketched as follows: I) Understand the problem, II) Think of a plan by solving subproblems, connection with older problems, etc. III) Walk along the plan while controlling each step. IV) Control the result, is the result obvious, can one use the method for other problems also?

The problem: Find the kernel and the image of A^{-1} if $A = \begin{pmatrix} 1 & 2 & 5 & 7 \\ 2 & 3 & 3 & 2 \\ 3 & 2 & 2 & 2 \\ 8 & 2 & 5 & 1 \end{pmatrix}$.

Can we solve a more general problem?

Solution: the more general problem is: assume a matrix is invertible, what is its kernel? We know that $Ax = 0$ has a unique solution $x = 0$. Therefore, the kernel is $\{0\}$. Also, because $Ax = b$ has always a unique solution $x = A^{-1}b$, the image of A is the entire four dimensional space.

Observation: the more general problem was easier to solve than the specific problem, inverting A and determining the kernel of A the usual way.

The problem: Find the kernel of the 4x4-matrix $B = \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}$, where A is the 3x3 matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 1 & 3 & 1 \end{pmatrix}$.

Did we see the objects of the problem in an earlier context?

Solution: on wednesday during a short discussion on fractals, we have seen that the matrix A has the property that $A(x, 1) = (Bx + b, 1)$. More generally, $A(x, r) = (Bx + rb, r)$. Now $A(x, r) = (0, 0)$ means that $r = 0$ and $Bx = 0$. So, the kernel of A is closely related to the kernel of B. The kernel of A consists of all vectors $(x, 1)$, where x is in the kernel of B. Since the kernel of B is the line containing $\{0, t, 0\}$, the kernel of A is the line containing $(0, t, 0, 0)$.

Observation: Having some older examples at hand helps to solve other problems.

The problem: Find the kernel and the image of $A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$, where A, B, C are $3x3$ matrices, whose kernel and images are known.

Can we solve a special case?

Solution: The simplest case is, where B, C, D are numbers. Then $Bx + Cy = 0, Dy = 0$ means $y = 0$ and so $Bx = 0$ or $x = 0$.
But this calculation can now be viewed also more generally: $Dy = 0$ means that y must be in the kernel of D and $Bx = 0$ means that x must be in the kernel of B. The kernel of A contains therefore all vectors (x, y), where x is in the kernel of C and y is in the kernel of D.

Observation: Sometimes, trying to solve a special case can lead to the solution in the general case.

The problem: Can we find the kernel of $(A - 1)$, where A is the rotation around the axis $(1, 2, 3)$ with angle 30 degrees?

What does the problem mean in an other language? What does it mean geometrically for example?

Solution: $(A - 1)x = 0$ means that $Ax = x$. This means x is left invariant under the transformation. Imagine a rotation. Which vectors are left invariant under the rotation?

Observation: Thinking about the problem geometrically helped to solve the problem without launching the analytic machinery.

The problem: Assume $ker(A) = ker(A^2)$. Is it true that $ker(A^3) = ker(A^2)$?

Can we try examples?

Solution: Try the case, when A is invertible. Try the case when $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Try the case, when A is a projection. Looks all good, lets try to prove it: assume x is in the kernel of A^3 but not in the kernel of A^2. This means $A^2x = y$ is not zero but $Ay = A^3x = 0$. But this means that $z = Ax$ is in the kernel of A^2 and so in the kernel of A so that $y = A^2x = Az = 0$ means that $y = 0$.

Observation: Especially in True/False problems, it can pay off to look at some specific examples before forming an opinion.