

Homework for Monday April 23, 1,2,3,4,5,6 in handout

SUMMARY. For linear systems $\dot{x} = Ax$ we can do a complete analysis of the system. The eigenvalues determine the behavior completely. For nonlinear systems, **chaos** can set in (i.e. the Lorentz system), explicit formulas for solutions are no more available in general (not even power series solutions which are valid for all t) and orbits can go off to infinity in finite time: example: $\dot{x} = x^2$ has the solution $x(t) = -1/(t - x(0))$. If $x(0) = 1$, we reach infinity at time $t = 1$.

How do nonlinear systems emerge? Linearity is often too crude. The exponential growth of $\dot{x} = ax$ of a bacteria colony for example is slowed down due to lack of food and the **logistic model** $\dot{x} = (a/M)x(M - x)$ is more accurate, where M is the population size at which bacteria starve to so much that the same number die then emerge: $x(t) = M$, then $\dot{x}(t) = 0$. Nevertheless, nonlinear systems can be investigated. But **qualitative methods** will become more important than **quantitative methods**. In two dimensions $\dot{x} = f(x, y), \dot{y} = g(x, y)$, where chaos does not happen, the analysis of **equilibrium points** and the **linear approximation** at those points in general allows to understand the system completely. In general, the analysis of equilibrium points and linear approximation at those points is a good starting point. Linear algebra plays an important role for investigating nonlinear systems too.

EQUILIBRIUM POINTS. A point x_0 is an **equilibrium point** of $\dot{x} = f(x)$ if $f(x_0) = 0$. If $x(0) = x_0$ then $x(t) = x_0$ for all times.

JACOBIAN MATRIX. If x_0 is an equilibrium point for $\dot{x} = f(x)$ then $[A]_{ij} = \frac{\partial f_i}{\partial x_j}(x_0)$ is called the **Jacobian** at x_0 . For two dimensional systems

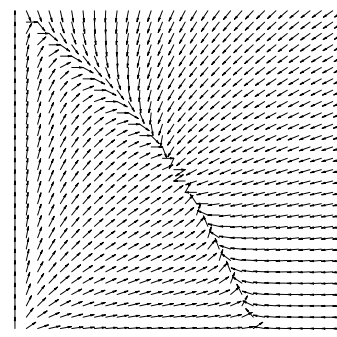
$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y) \end{aligned} \quad \text{this is the } 2 \times 2 \text{ matrix} \quad A = \begin{bmatrix} \frac{\partial f}{\partial x}(x, y) & \frac{\partial f}{\partial y}(x, y) \\ \frac{\partial g}{\partial x}(x, y) & \frac{\partial g}{\partial y}(x, y) \end{bmatrix}.$$

VECTOR FIELD. In two dimensions, we can draw the vector field by hand: attaching a vector $(f(x, y), g(x, y))$ at each point (x, y) . To find the equilibrium points it helps to draw the **nullclines** $\{f(x, y) = 0\}, \{g(x, y) = 0\}$. The equilibrium points are on intersections of nullclines. The eigenvalues of the Jacobians at equilibrium points allow to draw the vector field near equilibrium points. This information is sometimes enough to draw the vector field **by hand**, however schoolbooks avoid cases like $\dot{x} = x^5 + x^3y^2 - y, \dot{y} = x^7y^3 - y^4x - y^3x^2 - 1$, where it would be hard to draw isoclines and equilibrium points without technology.

MURRAY SYSTEM. $\dot{x} = x(6 - 2x - y), \dot{y} = y(5 - x - y)$ has the nullclines $x = 0, y = 0, 2x + y = 6, x + y = 5$. There are 4 equilibrium points $(0, 0), (3, 0), (0, 4), (2, 2)$. The Jacobian matrix of the system at the point (x_0, y_0) is

$$\begin{bmatrix} 6 - 4x_0 - y_0 & -x_0 \\ -y_0 & 5 - x_0 - 2y_0 \end{bmatrix}.$$

Equilibrium	Jacobian	Eigenvalues	Nature of equilibrium
(0,0)	$\begin{bmatrix} 6 & 0 \\ 0 & 4 \end{bmatrix}$	$\lambda_1 = 6, \lambda_2 = 4$	Unstable source
(3,0)	$\begin{bmatrix} -6 & -3 \\ 0 & 2 \end{bmatrix}$	$\lambda_1 = -6, \lambda_2 = 2$	Hyperbolic saddle
(0,4)	$\begin{bmatrix} 2 & 0 \\ -4 & -3 \end{bmatrix}$	$\lambda_1 = 2, \lambda_2 = -3$	Hyperbolic saddle
(2,2)	$\begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix}$	$\lambda_i = -3 \pm \sqrt{5}$	Stable sink



USING TECHNOLOGY (Example: Mathematica). Plot the vector field:

Needs["Graphics`PlotField`"]

f[x_, y_] := {x(6-2x-y), y(5-x-y)}; PlotVectorField[f[x, y], {x, 0, 4}, {y, 0, 4}]

Find the equilibrium solutions:

Solve[{x(6-2x-y)==0, y(5-x-y)==0}, {x, y}]

Find the Jacobian and its eigenvalues at (2, 2):

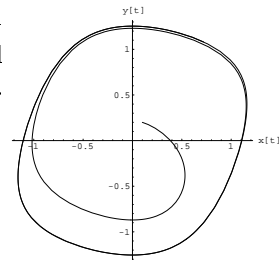
A[{x_, y_}] := {{6-4x, -x}, {-y, 5-x-2y}}; Eigenvalues[A[{2, 2}]]

Plotting an orbit:

S[{u_, v_}] := NDSolve[{x'[t] == x[t](6-2x[t]-y[t]), y'[t] == y[t](5-x[t]-y[t]), x[0] == u, y[0] == v}, {x, y}, {t, 0, 1}]

ParametricPlot[Evaluate[{x[t], y[t]} /. S[0.3, 0.5]], {t, 0, 1}, AspectRatio -> 1, AxesLabel -> {"x[t]", "y[t]"}]

EXAMPLE: LIENHARD SYSTEMS VAN DER POL EQUATION $\ddot{x} + (x^2 - 1)\dot{x} + x = 0$ appears in electrical engineering, biology or biochemistry. There $F(x) = x^3/3 - x, g(x) = x$.



$$\begin{aligned} \dot{x} &= y - F(x) & \dot{x} &= y - (x^3/3 - x) \\ \dot{y} &= -g(x) & \dot{y} &= -x \end{aligned}$$

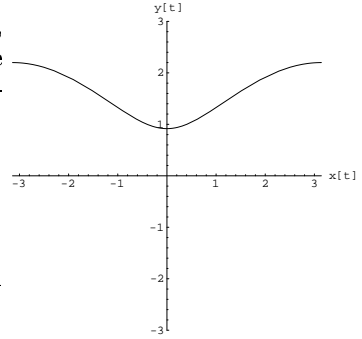
Lienhard systems have **limit cycles**. A trajectory always ends up on that limit cycle. This is useful for engineers, who need oscillators which are stable under changes of parameters. One knows: if $g(x) > 0$ for $x > 0$ and F has exactly three zeros $0, a, -a, F'(0) < 0$ and $F'(x) \geq 0$ for $x > a$ and $F(x) \rightarrow \infty$ for $x \rightarrow \infty$, then the corresponding Lienhard system has exactly one stable limit cycle.

EXAMPLE: HAMILTONIAN SYSTEMS are systems of the form

$$\begin{aligned} \dot{x} &= \partial_y H(x, y) & \dot{x} &= y \\ \dot{y} &= -\partial_x H(x, y) & \dot{y} &= -\sin(x) \end{aligned}$$

where H is called the **energy**. Usually, x is the position and y the momentum.

THE PENDULUM is an example, where $H(x, y) = y^2/2 - \cos(x)$ is the sum of the kinetic and potential energy.



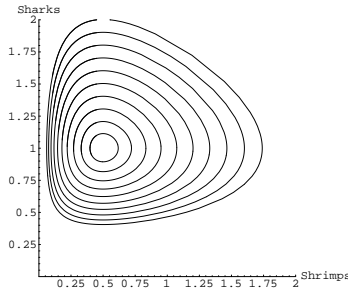
x is the angle between the pendulum and y-axis, y the angular velocity.

(More about the pendulum in the homework). Hamiltonian systems preserve energy $H(x, y)$ because $\frac{d}{dt}H(x(t), y(t)) = \partial_x H(x, y)\dot{x} + \partial_y H(x, y)\dot{y} = \partial_x H(x, y)\partial_y H(x, y) - \partial_y H(x, y)\partial_x H(x, y) = 0$. Orbits stay on level curves of H .

VOLTERRA-LODKA SYSTEMS are systems of the form

$$\begin{aligned} \dot{x} &= 0.4x - 0.4xy \\ \dot{y} &= -0.1 * y + 0.2xy \end{aligned}$$

This example has equilibrium points $(0, 0)$ and $(1/2, 1)$.

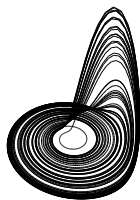


It describes for example shrimp-shark populations. The shrimp population $x(t)$ becomes smaller when there are more sharks, the shark population grows with more shrimps. (Volterra explained so first the oscillation of fish populations in the Mediterian sea).

CHAOS can occur for systems $\dot{x} = f(x)$ in three dimensions. Examples are 1D systems of the form $\ddot{x} = f(x, t)$, which becomes a three-dimensional system in the coordinates $(x, y, z) = (x, \dot{x}, t)$, systems $\ddot{x} = f(x, \dot{x})$ in two dimensions which becomes a four dimensional system in the coordinates (x, \dot{x}) or for systems $\dot{x} = f(x, t)$ in two dimensions. The term **chaos** has no uniform definition but usually means that one can find a true random number generator embedded in the system. Chaos theory is 100 years old. Basic insight had been obtained by Poincaré already. Since about 40 years, the subject exploded to a science, partly due to the availability of computers. Despite a lot of research, there are plenty of questions open.

THE ROESSLER SYSTEM is similar to the Lorentz system.

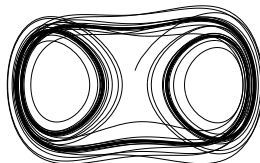
$$\begin{aligned} \dot{x} &= -(y + z) \\ \dot{y} &= x + 0.2 * y \\ \dot{z} &= 0.2 + x * z - 5.7 * z \end{aligned}$$



In three dimensions there is enough room to fold and mess things up: the attractor appears to ly on a band which gets stretched in one part of the phase space, then folded over onto itself. This "kneading of the dough" produces chaos.

THE DUFFING SYSTEM $\ddot{x} + \dot{x}/10 - x + x^3 - 12 \cos(t) = 0$ (metallic plate between magnets).

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -y/10 - x + x^3 - 12 \cos(z) \\ \dot{z} &= 1 \end{aligned}$$



Other chaotic examples can be obtained from mechanics like the **driven pendulum** $\ddot{x} + \sin(x) - \cos(t) = 0$.