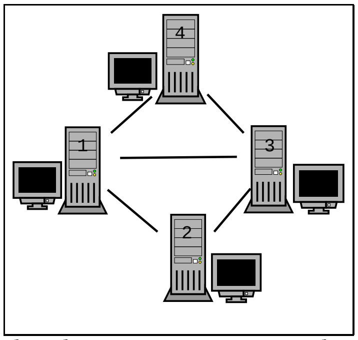
Lesson 7 Composing linear maps 2/20/2001 — Math 21b, O. Knill

Homework for Friday: 4,14,26*,28,40,48

Why do we need to compose linear transformations? Matrix multiplication is a generalisation of usual multiplication of numbers or the dot product. (Even the crossed product can be viewed as a matrix multiplication). In the three examples on this page, we see situations, where the composition of linear maps is essential. We will see later, that it helps to solve other problems like also to solve problems of the form $Ax = b$.



Networks, Connection and worms. Let us associate to the above computer network a matrix

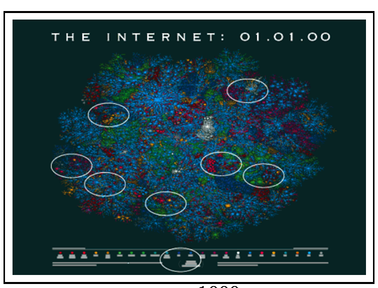
$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

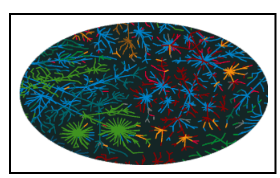
If a packet is at the first computer we associate with it a vector

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

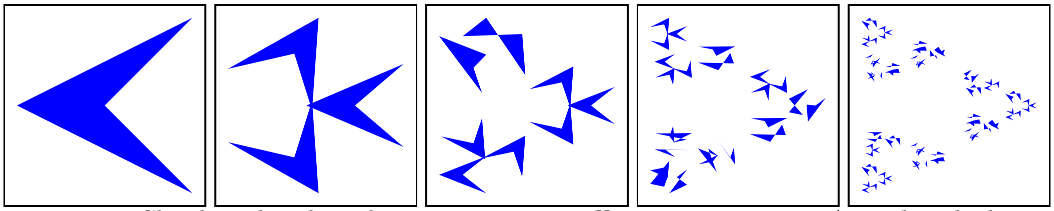
Now Ax labels the possible places with a 1, where the packet is in the next step. $(AA)(x)$ tells, in how many ways a packet can go from the first computer to the others in 2 steps. For example, it can go in three different ways back to the computer itself.

Using matrices is is a convenient way to solve combinatorial problems.



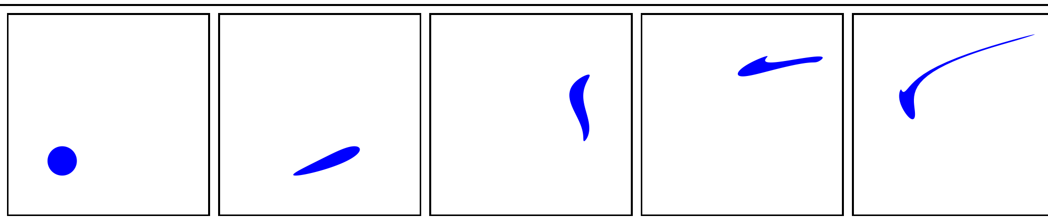


a) What does $[A^{1000}]_{22}$ tell about the network like the internet shown above as a graph?
b) Assume we have a local area network (LAN) and know that A^{100} does not contain any zero. What does it say about this computer network?
c) At computer Nr 2 of the above little, a worm (a la LoveBug Virus) is released. It travels each timestep to each neighboring computer. How many worms are at computer Nr. 3 in 4 time steps?



Fractals. Closely related to linear maps are **affine maps** $x \mapsto Ax + b$, which are compositions of a linear map with a translation. It is **not** a linear map because a linear map has the property that $B0 = 0$. However, one can write maps as linear maps in a space with one variable more by looking at $y = \begin{pmatrix} x \\ 1 \end{pmatrix}$ and the $(n+1) * (n+1)$ matrix $B = \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}$. Now, $By = \begin{pmatrix} Ax + b \\ 1 \end{pmatrix}$. We can model affine maps with linear maps. One construction of fractals goes as follows. Take for example 3 affine maps f, g, h which have the property to contract the space. (They can be obtained by composing rotations with dilations). For a given object Y_0 (like seen in the picture to the left), define now $Y_1 = f(Y_0) \cup g(Y_0) \cup h(Y_0)$ and recursively $Y_k = f(Y_{k-1}) \cup g(Y_{k-1}) \cup h(Y_{k-1})$. Above, we see what happens with the object after a few such iterations. In the limit, we obtain an object which is called a **fractal**. It has a dimension which is smaller than 2, the dimension of the plane but bigger than 1, the dimension of the line.

Challenge: Can you figure ou the three linear maps in the above picture?



Chaos. Chaologists study maps in the plane like for example

$$T : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2x + 2\sin(x) - y \\ x \end{pmatrix}$$

They iterate this map again anain again and look at $(x_1, y_1) = T(x, y), (x_2, y_2) = T(T(x, y))$, etc. One writes T^n for the n-th iteration of the map. and (x_n, y_n) for the image of (x, y) under the map T^n.

The linearisation of the map at a point (x, y) is the matrix

$$DT(x, y) = \begin{pmatrix} 2 + 2\cos(x) & -1 \\ 1 & \end{pmatrix} .$$

Note that $T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix}$, then the row vectors of $A(x, y)$ are just the gradients of f and g.

One calls the map T **chaotic at (x,y)**, if $DT^n(x, y)$, the linearisation of T^n grows exponentially fast with n. The **chain rule** tells that DT^n is the product of matrices $DT(x_i, y_i)$. If the points at which T is chaotic is large (in the sense that if one choses (x, y) randomly then we have a positive probability to have T chaotic there), then T is called chaotic.

Problem: Show that T is chaotic at $(0, 0)$. To see the pattern, look at a few examples $A = DT, DT^2 = AA, DT^3 = AAA$.