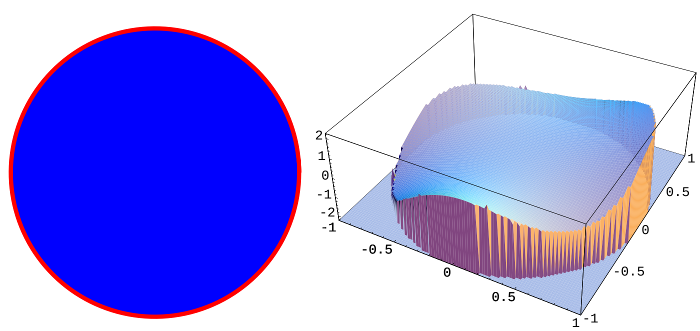
DIRICHLET PROBLEM: find a function $T(x, y)$ defined in a region Ω of the plane which satisfies $\Delta T(x, y) = 0$ inside the region and takes prescribed values $T_0(x, y)$ at the boundary. This boundary value problem can be solved explicitely in some cases.

PHYSICAL RELEVANCE.
- If the **temperature** distribution on the boundary of a region Ω, then $T(x, y)$ is the stable temperature distribution inside that region.

- If the **potential** of an electric charge distribution is given on the boundary of Ω and no charge is inside the region, then $U(x, y)$ satisfies $\Delta U = 0$. The electric field inside Ω is then ∇U.

An other application is that the velocity v of an ideal incompressible fluid satisfies $v = \nabla U$, where U satisfies the Laplace differential equation $\Delta U = 0$ in the region.

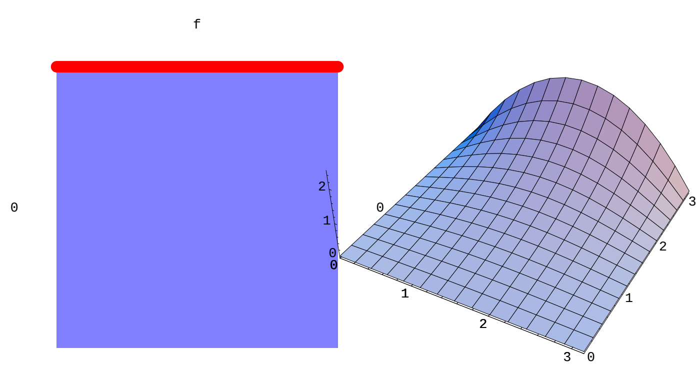
EXAMPLE: DISC.
In the unit disc $\{x^2 + y^2 \leq 1\}$, the problem is solved using complex analysis. Complex functions work well because if $f(z)$ is a complex function with a convergent Taylor series, then $(x, y) \mapsto \text{Re}(f)(x + iy)$ and $(x, y) \mapsto \text{Im}(f)(x + iy)$ are harmonic. For example $\text{Re}(z^2) = x^2 - y^2 + i(2xy)$ gives the two harmonic functions $f(x, y) = x^2 - y^2$ and $g(x, y) = 2xy$: they satify $(\partial_x^2 + \partial_y^2)f = 0$.
If one writes $z = x + iy = re^{i\theta}$ and $T(x, y) = f(z)$, then there is an explicity solution is given by the **Poisson integral formula**:



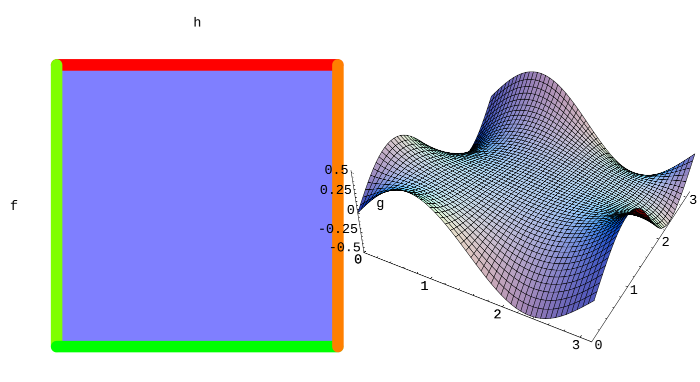
$$f(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{r^2 - 2r\cos(\theta - \phi) + 1} f(e^{i\phi})\, d\phi \,.$$

EXAMPLE: SQUARE.
We solve first the case, when T vanishes on three sides $x = 0, x = \pi, y = 0$ and $T(x, \pi) = f(x)$. Seperation of variables gives then $T(x, y) = \sum_n a_n \sin(nx) \sinh(ny)$, where the coefficients a_n are obtained from $T(x, \pi) = \sum_n b_n \sin(nx) = \sum_n a_n \sin(nx) \sinh(n\pi)$, where $b_n = \frac{2}{\pi} \int 0^\pi f(x) \sin(nx)\ dx$ are the Fourier coefficients of f. The solution is therefore

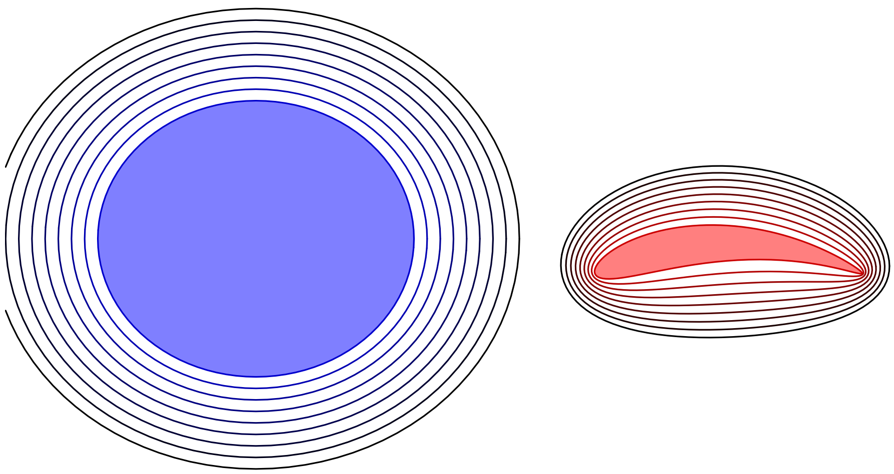
$$T_f(x, y) = \sum_{n=1}^{\infty} b_n \sin(nx) \sinh(ny) / \sinh(n\pi)$$



Solutions to general boundary conditions can be obtained by adding the solutions in the following cases: $T_f(x, \pi - y)$ solves the case, when T vanishes on the sides $x = 0, x = \pi, y = \pi$ and is f on $y = 0$. The function $T_f(y, x)$ solves the case, when T vanishes on the sides $y = 0, y = \pi, x = 0$ and $T_f(\pi - y, x)$ solves the case, when T vanishes on $y = 0, y = \pi$ and $x = \pi$. The general solution is $T(x, y) = T_f(x, y) + T_g(x, \pi - y)$
$+T_h(y, x) + T_k(\pi - y, x)$



A GENERAL REGION.
For a general region, one uses numerical methods. One possibility is by **conformal transportation**: to map the region into the the disc using a complex map F, which maps the region Ω into the disc. Solve then the problem in the disc with boundary value $T(F^{-1}(x,y))$ on the disc. If $S(x,y)$ is the solution there, then $T(x,y) = S(F^{-1}(x,y))$ is the solution in the region Ω.



The picture shows the example of the Joukowski map $z \mapsto (z + 1/z)/2$ which has been used in fluid dynamics (N.J. Joukowski (1847-1921) was a Russian airodynamics researcher.)
COMPLEX DYNAMICS. Also in complex dynamics, harmonic functions appear. Finding properties of complicated sets like the **Mandelbrot set** is done by mapping the exterior to the outside of the unit circle. If the Mandelbrot set is charged then the contour lines of equal potential can be obtained as the corresponding contour lines in the disc case (where the lines are circles).