The heat equation 5/2/2001 — Math 21b, O. Knill

Homework for Friday May 4, 1,2,3,4,5*,6* in Section 10.3

THE HEAT EQUATION. The temperature distribution $T(x,t)$ of a metal bar $[0,T]$ satisfies the **heat equation** $\dot{T}(x,t) = \mu T''(x,t)$. At every point, the rate of change of the temperature is proportional to the second derivative of $x \dots T(x,t)$ at x. The function $T(t,x)$ is zero at both ends of the bar and $f(x) = T(0,x)$ is a given initial temperature distribution.

SOLVING IT. This **partial differential equation** (PDE) is solved by writing $T(x,t) = u(x)v(t)$ which leads to $\dot{v}(t)u(x) = v(t)u''(x)$ or $\dot{v}(t)/(\mu v(t)) = u''(x)/u(x)$. Because the LHS does not depend on x and the RHS not on t, this must be a constant λ. We have seen already that $u''(x) = \lambda u(x)$ is an eigenvalue problem $D^2 u = \lambda u$ which has only solutions when $\lambda = -k^2$ for integers k. The eigenfunctions of D^2 are $u_k(x) = \sin(kx)$. The second equation $\dot{v}(t) = k^2 \mu v$ is solved by $v(t) = e^- k^2 \mu t$ so that $u(x,t) = \sin(kx)e^{-k^2 t}$ is a solution of the heat equation. Because linear combinations of solutions are solutions too, we obtain solution is of the form $T(x,t) = \sum_{k=1}^{\infty} a_k \sin(kx) e^{-k^2 t}$ which is for fixed t a **Fourier series**. The coefficients a_k are determined from $f(x) = T(x,0) = \sum_{k=1}^{\infty} a_k \sin(kx)$, where $a_k = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(kx)$. (The factor 2 comes from the fact that f should be thought to be extended to an odd function on $[-\pi,\pi]$ and the integral from $[-\pi,0]$ is the same as the from $[0,\pi]$.)

FOURIER:
(birth of Fourier theory)

> The heat equation $\dot{T}(t,x) = \mu T''(t,x)$ with smooth $T(x,0) = f(x), T(0,0) = T(\pi,0) = 0$ has the solution $\sum_{k=1}^{\infty} a_k \sin(kx) e^{-k^2 \mu t}$ with $a_k = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(kx)\, dx$.

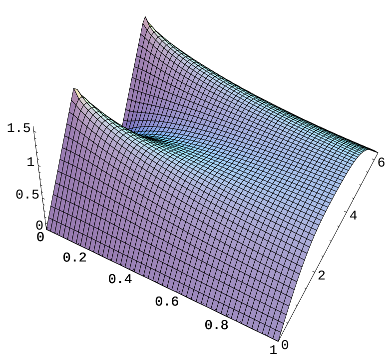
Remark. If $T(0,0) = a, T(\pi,0) = b$ instead, then the solution is $T(x,t) = T_0(x,t) + a + bx/\pi$, where $T_0(x,t)$ is the solution in the box using $f_0(x) = f(x) - (a + bx/\pi)$ which satisfies $f(0) = f(\pi) = 0$. (Because $T(t,x) = a + bx/\pi$ is a solution of the heat equation, we can add it to any solution and get new solutions. This allows to tune the boundary conditions).

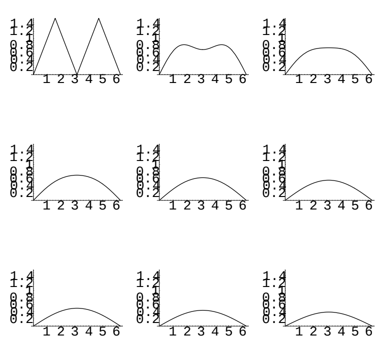
EXAMPLE. $f(x) = x$ on $[-\pi/2, \pi/2$ π-periodically continued has the Fourier coefficients $a_k = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} x \sin(kx)\, dx = 4/(k^2\pi)(-1)^{(k-1)/2}$ for odd k and 0 else. The solution is

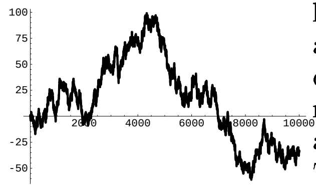
$$T(x,t) = \sum_{m=0}^{\infty} \frac{4(-1)^m}{(2m+1)^2 \pi} e^{-\mu(2m+1)^2 t} \sin((2m+1)x) \; .$$

The exponential term containing the time makes the function $T(x,t)$ converge to 0: "The bar cools." The higher frequency terms are damped faster because "Smaller disturbances are smoothed out faster."

VISUALIZATION. We can just plot the graph of the function $T(x,t)$ or plot the temperature distribution for different times t_i.







DERIVATION OF THE HEAT EQUATION. The temperature T measures the kinetic energy distribution of atoms, and electrons in the bar. Each particle makes a **random walk**. We can model that as follows: assume we have n adjacent cells containing particles and in each time step, a fraction of the particles moves randomly either to the right or to the left. If $T_i(t)$ is the energy of particles in cell i at time t, then the energy of particles at time $t+1$ is $\frac{\rho}{2}(T_{i-1}(t) - 2T_i(t) + T_{i+1})(t))$. This is a discrete version of the second derivative $T''(x,t) \sim (T(x+dx,t) - 2T(x,t) + T(x+dx,t))/(dx^2)$.

FOURIER TRANSFORM=DIAGONALISATION OF D. The Fourier transform U actually **diagonalizes** the linear map D: write $f = \sum_n \hat{f}_n e^{inx} = S^{-1}f$ and $Sf = \hat{f} = (\dots, c_{-1}, c_0, c_1, \dots)$. $SDS^{-1}\hat{f} = SD\sum_n \hat{f}_n e^{inx} = S\sum_n n\hat{f}_n e^{inx} == (\dots, i(-1)c_{-1}, 0c_0, ic_1, \dots)$. This means that differentiation changes the Fourier coefficients as $c_n \mapsto inc_n$. From this we see that the diagonalisation of D^2 is multiplying with $-n^2$. Now, if $\dot{u} = D^2u$, then $\dot{\hat{u}}_n = -\mu n^2 \hat{u}_n$ so that $\hat{u}_n(t) = e^{-n^2t}\hat{u})n(0)$. The linear Fourier transform brings the partial differential equation into a sequence of ordinary differential equations.

Differentiation of the function

$$\begin{array}{ccc} f & \overset{S}{\leftarrow} & \hat{f} \\ D\downarrow & & \downarrow M \\ Df & \overset{S^{-1}}{\rightarrow} & M\hat{f} \end{array}$$

becomes multiplication with the diagonal matrix

$$M = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 2i & 0 & 0 & 0 & 0 & \cdot \\ \cdot & 0 & i & 0 & 0 & 0 & \cdot \\ \cdot & 0 & 0 & 0 & 0 & 0 & \cdot \\ \cdot & 0 & 0 & 0 & i & 0 & \cdot \\ \cdot & 0 & 0 & 0 & 0 & 2i & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

QUANTUM MECHANICS = MATRIX MECHANICS. Quantum mechanics had been formulated in different ways. It has been realized by P.Jordan that these formulations are equivalent. Remember the particle in a box? It was discribed by the linear map $L = -D^2\hbar^2/(2m)$ acting on functions on $[0,\pi]$. In the Fourier picture, this linear map is

$$M = \frac{\hbar^2}{2m}\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -4 & 0 & 0 & 0 & 0 & \cdot \\ \cdot & 0 & -1 & 0 & 0 & 0 & \cdot \\ \cdot & 0 & 0 & 0 & 0 & 0 & \cdot \\ \cdot & 0 & 0 & 0 & -1 & 0 & \cdot \\ \cdot & 0 & 0 & 0 & 0 & -4 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

The diagonal entries are the eigenvalues of L as it always the case after diagonalisation. The linear map L is actually "symmetric" which - as in finite dimensions - assures that the matrix can be diagonlized with real entries in the diagonal.

Quantum mech. observables are symmetric linear maps: position, $X: f \mapsto xf$, momentum $P: f \mapsto i\hbar\nabla f$, kinetic energy $f \mapsto P^2/2m = -\hbar^2\Delta/(2m)$, Coulomb potential energy $f \mapsto (4\pi\epsilon_0)^{-1}e^2/rf$.

COMPARISON WITH THE WAVE EQUATION. The wave equation $\ddot{T}(x,t) = c^2T''(x,t)$ is solved in a similar way as the heat equation: writing $f(x,t) = u(x)v(t)$ gives $\ddot{v}u = c^2vu''$ or $\ddot{v}/(c^2v) = u''/u$. Because the left hand is independent of x and the right hand side is independent of t, we have $\ddot{v}/(c^2v) = u''/u = -k^2 = const$. The right equation has solutions $u_k(x) = \sin(kx)$. Now, $v_k(t) = a_k\cos(kct) + b_k\sin(kct)$ solves $\ddot{v} = -c^2k^2v$ so that $T_k(x,t) = u_k(x)v_k(t) = \sin(kx)(a_k\cos(kct) + b_k\sin(kct))$ is a solution of the wave equation. General solutions can be obtained by taking **superpositions of these waves** $T(x,t) = \sum_{k=1}^\infty a_k(\sin(kx)\cos(kct) + b_k\sin(kx)\sin(kct)$. The coefficients a_k, b_k are obtained from $T(x,0) = \sum_{k=1}^\infty a_k\sin(kx)$ and $\dot{T}(x,0) = \sum_{k=1}^\infty kcb_k\sin(kx)$. Unlike in the heat equation both $T(x,0)$ and $\dot{T}(x,0)$ have to be specified.

WAVE EQUATION:

The wave equation $\ddot{T}(t,x) = c^2T''(t,x)$ with smooth $T(x,0) = f(x), \dot{T}(x,0) = g(x), T(0,0) = T(\pi,0) = 0$ has the solution $\sum_{k=1}^\infty a_k\sin(kx)\cos(kct) + b_k\sin(kx)\sin(kct)$ with $a_k = \frac{2}{\pi}\int_0^\pi f(x)\sin(kx)\,dx$ and $kcb_k = \frac{2}{\pi}\int_0^\pi g(x)\sin(kx)\,dx$.

HIGHER DIMENSIONS. The heat or wave equations can also be solved **in higher dimensions** using Fourier theory. For example, on a matellic plate $[-\pi,\pi] \times [-\pi,\pi]$ the temperature distribution $T(x,y,t)$ satisfies $T_t = \mu(T_{xx} + T_{yy})$.

The Fourier series of a function $f(x,y)$ in two variables is $\sum_{n,m} c_{n,m}e^{inx+my}$, where $c_{n,m} = (2\pi)^{-2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)e^{-inx}e^{-imy}\,dx\,dy$.

The Gaussian blur filter applied to each color coordinate of a picture acts very similar than the heat flow.