Continuous Dynamical Systems 4/16/2001	Math 21b, O. Knill

Homework for Wednesday April 18, 2,14*,26,32,46*,54

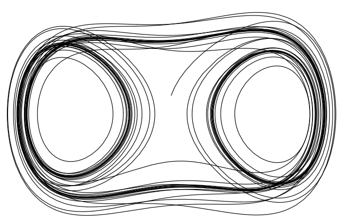
CONTINUOUS DYNAMICAL SYSTEMS. A differential equation $\dot{x} = f(x)$ defines a dynamical system $t \mapsto x(t)$. The solutions is a curve $x(t)$ which has the velocity vector $f(x(t))$ at each time t.

ONE DIMENSION. A system $\dot{x} = g(x,t)$ (written in the form $\dot{x} = g(x,t), \dot{t} = 1$) can be explicitely solved in examples like

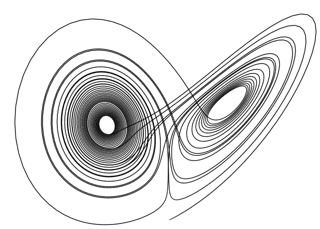
- If $\dot{x} = g(t)$, then $x(t) = \int_0^t g(t)\, dt$.
- If $\dot{x} = h(x)$, then $dx/h(x) = dt$ and so $t = \int_0^x dx/h(x) = H(x)$ so that $x(t) = H^{-1}(t)$.
- If $\dot{x} = g(t)/h(x)$, then $H(x) = \int_0^x h(x)\, dx = \int_0^t g(t)\, dt = G(t)$ so that $x(t) = H^{-1}(G(t))$.

In general there are no closed form solutions **in terms of known functions**. The system $\dot{x} = e^{-t^2}$ is an example, where $x(t) = \int_0^t e^{-t^2}\, dt$ can not be expressed in terms of functions exp, sin, log, $\sqrt{\cdot}$, *etc.*. However, $x(t)$, the anti-derivative of e^{-t^2} exists and can be written as a Taylor series ($e^{-t^2} = 1 - t^2 + t^4/2! - t^6/3! + \dots$ and taking in each term the anti-derivatives gives: $x(t) = t - t^3/3 + t^4/(32!) - t^7/(73!) + \dots$.

HIGHER DIMENSIONS. In higher dimensions, **chaos can set in** and the system can become unpredictable.



The nonlinear **Lorentz system** to the right $\dot{x}(t) = 10(y(t) - x(t))$, $\dot{y}(t) = -x(t)z(t) + 28x(t) - y(t)$, $\dot{z}(t) = x(t) * y(t) - 8z(t)/3$ shows a "strange attractor". Evenso completely **deterministic**: (from $x(0)$ all the path $x(t)$ is determined), there are observables which can be used as a random number generator. The **Duffing system** $\ddot{x} + \dot{x}.10 - x + x^3 - 12\cos(t) = 0$ to the left can be written in the form $\dot{v} = f(v)$ with a vector $v = (x, \dot{x}, t)$.

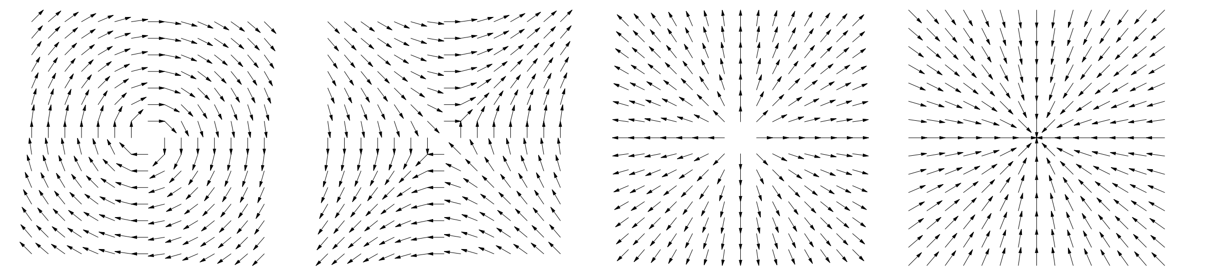


1D LINEAR DIFFERENTIAL EQUATIONS. A linear differential equation in one-dimension has the form $\dot{x} = \lambda x$. Its solution is $x(t) = e^{\lambda t} x(0)$. This differential equation appears in applications:

- As **population models** for $\lambda > 0$: the rate of births in a popolation is proportional to the number of people.
- As a model for **decay in a radioactive sample** for $\lambda < 0$: the rate of decay is proportional to the number of atoms.

LINEAR DIFFERENTIAL EQUATIONS IN HIGHER DIMENSIONS. Linear dynamical systems have the form $\dot{x} = Ax$, where A is a matrix. Such dynamical systems have the orgin 0 as an **equilibrium point**. The general solution is $x(t) = e^{At}$, where $e^{At} = 1 + At + A^2t^2/2! + \dots$ because if we differentiate this with respect to t, we obtain $\dot{x}(t) = A + 2A^2t/2! + \dots = A(1 + At + A^2t^2/2! + \dots) = Ae^{At} = Ax(t)$. If $B = S^{-1}AS$ is diagonal with the eigenvalues $\lambda_j = a_j + ib_j$ in the diagonal, then $y = S^{-1}x$ satisfies $y(t) = e^{Bt}$ and therefore $y_j(t) = e^{\lambda_j t} y_j(0) = e^{a_j t} e^{i b_j t} y_j(0)$. One gets the solutions in the original coordinates as $x(t) = Sy(t)$.

PHASE PORTRAITS. For differential equations $\dot{x} = f(x)$ in two dimensions one can **draw the vector field** $x \mapsto f(x)$. The solution $x(t)$ is tangent to the vector $f(x(t))$ at each time. From staring at the phase portraits and drawing some solution curves, one can get already a picture about the behavior of the system. Some examples of phase portraits of linear two-dimensional systems.



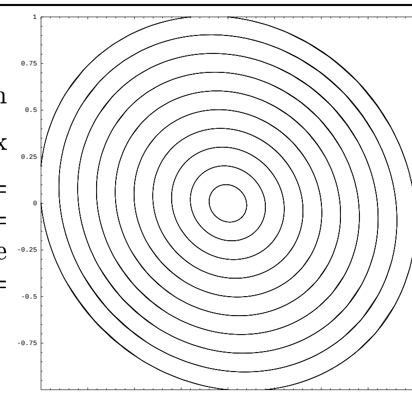
UNDERSTANDING A DIFFERENTIAL EQUATION. Having a closed form solution like $x(t) = e^{At}x(0)$ for $\dot{x} = Ax$ is actually quite useless in general. One wants to understand the solution quantitatively. Questions one wants to answer are: what happens in the long term? What is the stability of equilibrium solutions or periodic solutions. Can one decompose the system into simpler subsystems? For linear continuous systems, **diagonalisation does the job** and one really **understands the system:** diagonalisation decomposes the system into one-dimensional linear systems, which can be analyzed seperately. In general, "understanding" can mean different things: for example

Plotting phase portraits. Computing solutions numerically and estimate the error. Finding special solutions. Predicting the shape of some orbits. Finding regions which are invariant.	Finding special closed form solutions $x(t)$. Finding a power series $x(t) = \sum_n a_n t^n$ in t. Finding quantities which are unchanged along the flow (called "Integrals"). Finding quantities which increase along the flow (called "Lyapunov functions").

LINEAR STABILITY. A linear dynamical system $\dot{x} = Ax$ with diagonalizable A is linearly stable if and only if $a_j = \mathrm{Re}(\lambda_j) < 0$ for all eigenvalues λ_j of A.
PROOF. We see that from the explicit solutions $y_j(t) = e^{a_j t}e^{ib_j t}y_j(0)$ in the basis consisting of eigenvectors. Now, $y(t) \to 0$ if and only if $a_j < 0$ for all j and $x(t) = Sy(t) \to 0$ if and only if $y(t) \to 0$.

RELATION WITH DISCRETE TIME SYSTEMS. From $\dot{x} = Ax$, we obtain $x(t+1) = Bx(t)$, with the matrix $B = e^A$. The eigenvalues of B are $\mu_j = e^{\lambda_j}$. Clearly $|\mu_j| < 1$ if and only if $\mathrm{Re}\lambda_j < 0$. The criterium for linear stability of discrete dynamical systems is compatible with the criterium for linear stability of $\dot{x} = Ax$.

EXAMPLE 1. The system $\dot{x} = y, \dot{y} = -x$ can in vector form $v = (x, y)$ be written as $\dot{v} = Av$, with $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The matrix A has the eigenvalues $i, -i$. After a coordinate transformation $w = S^{-1}v$ we get with $w = (a, b)$ the differential equations $\dot{a} = ia, \dot{b} = -i = b$ which has the solutions $a(t) = e^{it}a(0), b(t) = e^{-it}b(0)$. The original coordates satisfy $x(t) = \cos(t)x(0) - \sin(t)y(0)$, $y(t) = \sin(t)x(0) + \cos(t)y(0)$. Indeed e^{At} is a rotation in the plane.



EXAMPLE 2. A **harmonic oscillator** $\ddot{x} = -x$ can be written with $y = \dot{x}$ as $\dot{x} = y, \dot{y} = -x$ (see Example 1). The general solution is therefore $x(t) = \cos(t)x(0) - \sin(t)\dot{x}(0)$.

EXAMPLE 3. We take **two harmonic oscillators and couple them**: $\ddot{x}_1 = -x_1 - \epsilon(x_2 - x_1), \ddot{x}_2 = -x_2 + \epsilon(x_2 - x_1)$. (for small $|x_i|$ one can simulate this with two coupled penduli which are connected with a spring). The system can be written as $\ddot{v} = Av$, with $A = \begin{bmatrix} -1+\epsilon & -\epsilon \\ -\epsilon & -1+\epsilon \end{bmatrix}$. The matrix A has an eigenvalue $\lambda_1 = -1$ to the eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and an eigenvalue $\lambda_2 = -1 + 2 * \epsilon$ to the eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The coordinate change S is $S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. It has the inverse $S^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}/2$. In the coordinates $w = S^{-1}v = (y_1, y_2)$, we have oscillations $\ddot{y}_1 = -y_1$ corresponding to the case $x_1 - x_2 = 0$ (the pendula swing synchroneous) and $\ddot{y}_2 = -(1-2\epsilon)y_2$ corresponding to $x_1 + x_2 = 0$ (the pendula swing against each other).