Determinant III,(last sequel of a trilogy) 3/19/2001 Math 21b, O.Knill

Homework for Wednesday, March 21, Nr. 2,4*,12,18*,22,32

AIM. We want explicit formulas for the inverse of a matrix A or the solution x of a linear equation $Ax = b$. While writing this in terms of determinants is **not** the most efficient way to compute these things, such formulas are useful for example when having parameters in the matrix. A symbolic algebra program can then for example give explicit formulas for the dependence of the solution x on external parameters and allow theoretical predictions.

REMINDERS.
- An orthonormal matrix satisfies $Q^TQ = 1$. From this we get $\det(Q)^2 = 1$ and so $|\det(Q)| = 1$.
- The determinant of a tridiagonal matrix is the product of the diagonal entries.
- Every matrix can be written as $A = QR$, where Q is orthogonal and R is upper tridiagonal.
- The image of the unit cube under a linear map $x \mapsto Ax$ is a parallelepiped E_n spanned by the column vectors $v_1, \dots, v_n$ of A.

VOLUME OF A PARALLELEPIPED. A j-dimensional parallelepiped E_j spanned by vectors $v_1, ..., v_j$ has a $j-1$ dimensional parallelepiped E_{j-1} in the base. E_{j-1} is contained in the vector space V_{j-1} spanned by $v_1, \dots, v_{j-1}$. The opposite "face" is in distance $||u_j|| = ||v_j - \text{proj}_{V_{j-1}}||$. The volume $\text{vol}(E_j)$ satisfies $\text{vol}(E_{j-1})||u_j||$.

ORIENTATION. Determinants allow us to **define** the orientation of n vectors in n-dimensional space, (where we don't have a "right hand rule" in general ...). Just look at the matrix A with column vectors v_j and define the orientation as the sign of $\det(A)$. In three dimensions, this agrees with the right hand rule: if v_1 is the thumb, v_2 is the pointing finger and v_3 is the middle finger, then their orientation is positive.

DETERMINANT AND VOLUME. The absolute value of the determinant of a $n \times n$ matrix A is the volume of the n-dimensional parallelepiped E_n spanned by the column vectors v_j of A.
Proof. Use the QR decomposition $A = QR$, where Q is orthogonal and R is upper triangular. From $QQ^T = 1$, we get $1 = \det(Q)\det(Q^T) = \det(Q)^2$ see that $|\det(Q)| = 1$. Therefore, $\det(A) = \det(R)$. The determinant of R is the product of the $||u_i|| = ||v_i - \text{proj}_{V_{j-1}} v_i||$ which was the distance from v_i to V_{j-1}. The volume $\text{vol}(E_j)$ of a j-dimensional parallelepiped E_j with base E_{j-1} in V_{j-1} and height $||u_i||$ is $\text{vol}(E_{j-1})||u_j||$. Inductively $\text{vol}(E_j) = ||u_j||\text{vol}(E_{j-1})$ and therefore $\text{vol}(E_n) = \prod_{j=1}^n ||u_j|| = \det(R)$.

MORE GENERALLY: The volume of a k dimensional parallelepiped defined by the vectors $v_1, \dots, v_k$ is $\sqrt{\det(A^TA)}$ because $A^TA = (QR)^T(QR) = R^TR$ and $\det(R^TR) = \det(R)^2 == (\prod_{j=1}^n ||u_j||)^2$.

CHANGE OF VARIABLES. (For people who heard multi-variable calculus) If $x \mapsto y = u(x)$ is a change of variable, then the matrix $Du(x)$ is the linearisation of the map near x and $|dy| = ||\det(Du(x))|| \cdot |dx|$. This leads to the change of variable formula

$$\int_S f(x)\ dx = \int_{u(S)} f(y)\ |\det(Du^{-1}(y)|dy$$

which can be remembered like in the 1 dimensional case:

Example 1: (1 dim) if $x = u^{-1}(y) = \sin(y)$, $dx = \cos(y)\ dy$. $\int_0^1 \sqrt{1-x^2}\ dx = \int_0^{\pi/2} \sqrt{(1-\sin^2(y))}\ \cos(y)\ dy = \int_0^{\pi/2} \cos^2(y)\ dy = \pi/4$.
Example 2: If $u(s,t) = (x(s,t), y(s,t), x(s,t))$ is a surface, then $A = Du(s,t)$ is a $3x2$ matrix with column vectors $X = \begin{bmatrix} x_u \\ y_u \\ z_u \end{bmatrix}$ $Y = \begin{bmatrix} x_v \\ y_v \\ z_v \end{bmatrix}$ (which are tangent vectors to the surface). Now $A^TA = \begin{bmatrix} X \cdot X & X \cdot Y \\ X \cdot Y & Y \cdot Y \end{bmatrix}$ whose determinant is $||X||^2||Y||^2 - ||X \cdot Y||^2 = ||X||^2||Y||^2(1-\cos(\phi)^2) = ||X||^2||Y||^2 \sin(\phi)^2 = |X \times Y|^2$. The expansion factor is $|X \times Y|$.

CRAMER'S RULE. This is an explicit formula for the solution of $Ax = b$. If A_i is the matrix, where the column v_i is replaced by b, then $x_i = \det(A_i)/\det(A)$.
Proof. $\det(A_i) = \det([v_1, ..., b, ..., v_n] = \det([v_1, ..., (Ax), ..., v_n] = \det([v_1, ..., \sum_i x_i v_i, ..., v_n] = x_i \det([v_1, ..., v_n]) = x_i \det(A)$.

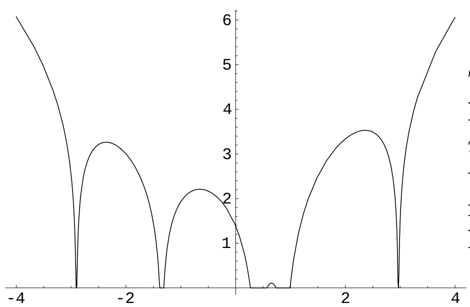
GABRIEL CRAMER. (1704-1752). Born in Geneva (Switzerland), he worked on geometry and analysis. He died during a trip to France, where he wanted to start retirement.

WHY IS CRAMERS RULE INTERESTING? Determining x with these formulas is slower than with Gaussian elmination: a determinant calculation needs n^3 steps so that n^4 calculations are needed for the inverse via Cramer's rule. (Compare n^3 with Gaussian elmination).
The rule is important because if A or b depends on a parmeter λ, and we want to see how x depends on the parameter λ one can find explicit formulas for $(d/d\lambda)x_i(\lambda)$.
Cramers rule tells for example that the solution can depends in a sensitive way on parameters if the determinant is small (look at scissors).

EXAMPLE. In solid state physics, one is interested in the $\det(L - E)$, where

$$L = \begin{bmatrix} \lambda\cos(\alpha) & 1 & 0 & \cdot & 0 & 1 \\ 1 & \lambda\cos(2\alpha) & 1 & \cdot & \cdot & 0 \\ 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 0 \\ 0 & \cdot & \cdot & 1 & \lambda\cos((n-1)\alpha) & 1 \\ 1 & 0 & \cdot & 0 & 1 & \lambda\cos(n\alpha) \end{bmatrix}$$

describes an electron in a periodic crystal, E is the energy and $\alpha = 2\pi/n$. The electron can move (as a Bloch wave) whenever the determinant is negative. These intervals form the **spectrum** of the matrix. A physist is interested for example in the dependence of the spectrum on the parameter λ or E.



The graph shows the function $E \mapsto \log(|\det(L - E)|)$ in the case $\lambda = 2$ and $n = 5$. In the energy intervals, where this function is zero, the electron can move, otherwiese the crystal is an insulator.

THE INVERSE OF A MATRIX. Because the columns of A^{-1} are solutions of $Ax = e_i$ with $e_i = \begin{bmatrix} 0 \\ \cdot \\ 1 \\ \cdot \\ 0 \end{bmatrix}$, Cramers rule together with the Lagrange expansion gives $A^{-1}_{ij} = e_j \cdot A^{-1} e_i = e_j \cdot x_j = (-1)^{i+j} \det(A_{ji})/\det(A)$.
The matrix $B_{ij} = (-1)^{i+j}\det(A_{ji})$ is called the **classical adjoint** of A. NOTE the change $ij \to ji$.
DON'T confuse the classical adjoint with the **transpose** A^T which is sometimes also called the **adjoint**.