

Homework for Friday March 16: Section 5.1, 16,24,26,36,48,52

WHY DO WE CARE ABOUT DETERMINANTS?

- check invertibility of matrices
- have geometric interpretation as volume
- some formulas involve determinants. i.e. in integration theory, or to invert a matrix.

PERMUTATIONS. A **permutation** of n elements $\{1, 2, \dots, n\}$ is a rearrangement of $\{1, 2, \dots, n\}$. There are $n! = n(n-1)\dots 1$ different permutations of $\{1, 2, \dots, n\}$ because we have n choices to choose the first element, $(n-1)$ choices to choose the second, etc until we reach the end, where we are forced to put the last one.

PATTERNS. The "graphs" $i, \pi(i)$ are called **patterns**. See all the possible 120 patterns in the case $n = 5$.

EXAMPLE. There are 6 permutations of $\{1, 2, 3\}$. They are

$$(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1).$$

SIGN. The sign of a permutation π , denoted by $(-1)^\pi$ is (-1) if one needs an odd number of transpositions (flips of two elements) to produce the permutation. Otherwise, the sign is $+1$.

EXAMPLE. The permutation $(1, 2)$ has sign $+1$, the permutation $(2, 1)$ has sign -1 .

The permutations $(1, 2, 3), (3, 2, 1), (2, 3, 1)$ have sign 1 the permutatations $(1, 3, 2), (3, 2, 1), (2, 1, 3)$ have sign -1 .

DEFINITION. The **determinant** of a $n \times n$ matrix A is the sum $\sum_{\pi} (-1)^\pi A_{1\pi(1)} A_{2\pi(2)} \dots A_{n\pi(n)}$, where π runs over all permutations of $\{1, 2, \dots, n\}$ and $\pi(n)$ is the sign of the permutation.

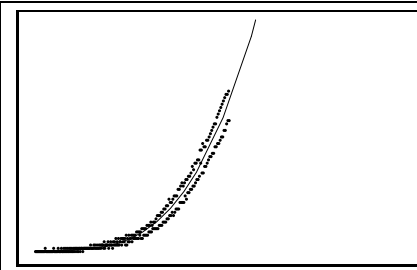
EXAMPLE 2×2 . The determinant of $A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is $ad - bc$. There are two permutations of $(1, 2)$. The identity permutation $(1, 2)$ gives $A_{11}A_{22}$, the permutation $(2, 1)$ gives $A_{21}A_{12}$. You might remember from Math21a, that $\det(A)$ is the area of the parallelogram spanned by the column vectors of A . The two vectors form a basis if and only if $\det(A) \neq 0$.

EXAMPLE 3×3 . The determinant of $A = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ is $aei + bfg + cdh - ceg - fha - bdi$ corresponding to the 6 permutations of $(1, 2, 3)$. You might remember that $\det(A)$ is the volume of the parallelepiped spanned by the column vectors of A . The three vectors form a basis if and only if $\det(A) \neq 0$.

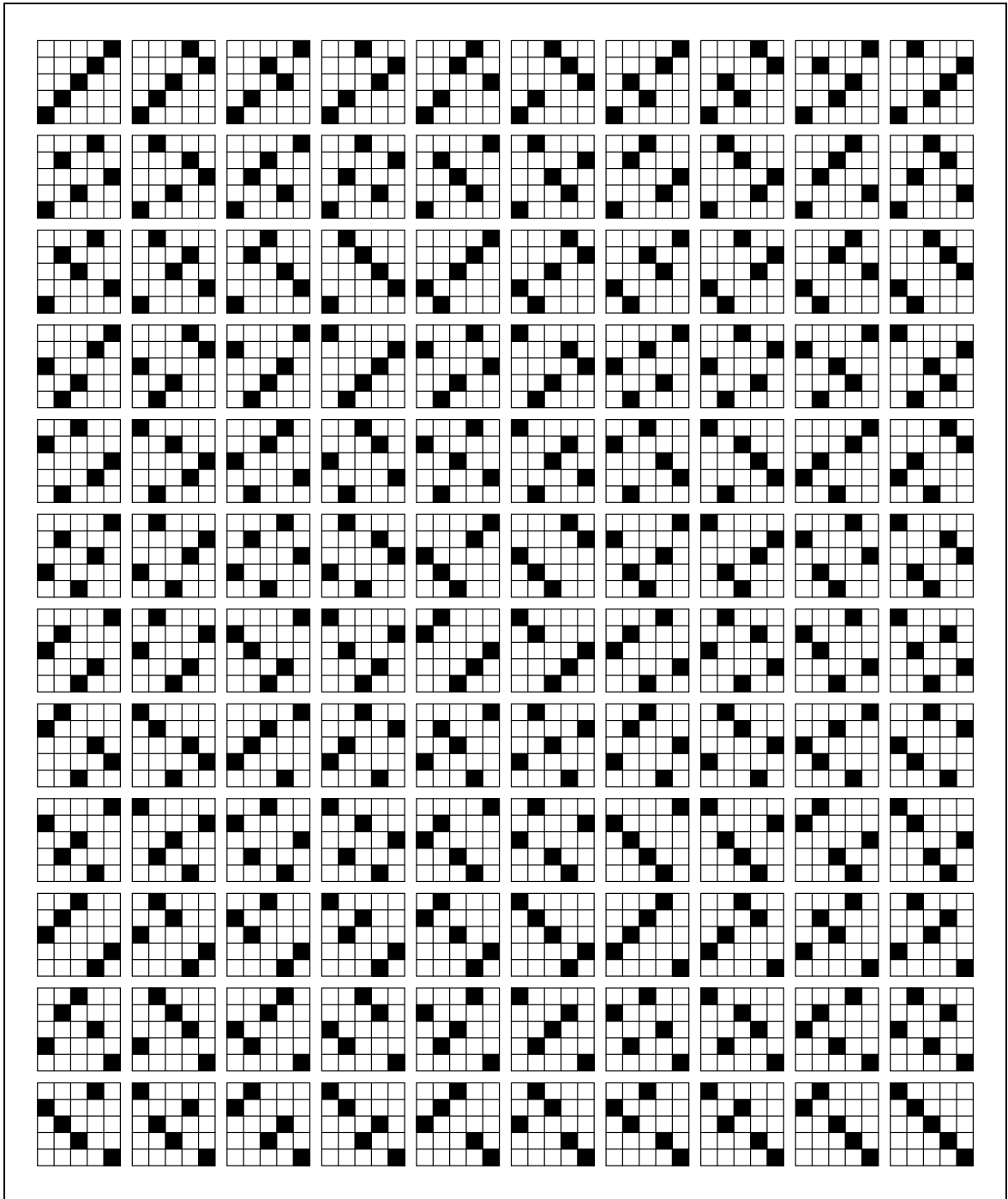
DIAGONAL MATRICES. The determinant of a diagonal matrix is the product of the diagonal elements.

TRIANGULAR MATRICES. The determinant of an upper tridiagonal matrix is the product of the diagonal elements.

HOW FAST CAN WE COMPUTE THE DETERMINANT?. Calculating the determinant can be done by done using Gauss Jordan elimination. Just multiply all the scaling factors during the elimination with the product of the diagonal elements of $\text{rref}(A)$. The cost is about the same as the cost for the Gauss-Jordan elimination.



The graph to the left shows some measurements of the time needed for Mathematica to calculate the determinant in dependence on the size of the $n \times n$ matrix. The matrix size goes from $n=1$ to $n=300$. We also see a best cubic fit of these data using the least square method from the last lesson.



All the $5!=120$ permutations of $\{1, 2, 3, 4, 5\}$.