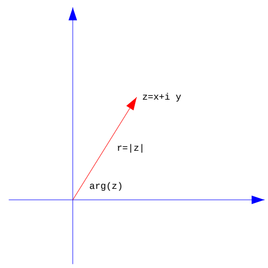
Complex Numbers, Complex Eigenvalues 4/2/2001	Math 21b, O. Knill

Homework for Friday, April 6, 2001: Section 6.4, 6,12,24,30*,36,38*



NOTATION. Write $z = x + iy = r\exp(i\phi) = r\cos(\phi) + ir\sin(\phi)$. The number $r = |z|$ is called the **absolute value** of z, the number ϕ is called the **argument** and denoted by $\arg(z)$. Complex numbers contain the **real numbers** $z = x + i0$ as a subset. One writes $\mathrm{Re}(z) = x$ and $\mathrm{Im}(z) = y$ if $z = x + iy$.

ARITHMETIC. Complex numbers are added like vectors: $x+iy+u+iv = (x+u)+i(y+v)$ and multiplied as $z * w = (x+iy)(u+iv) = xu - yv + i(yu - xv)$. If $z \neq 0$, one can divide $1/z = 1/(x+iy) = (x-iy)/(x^2+y^2)$.

ABSOLUTE VALUE AND ARGUMENT. The absolute value $|z| = \sqrt{x^2+y^2}$ satisfies $|zw| = |z|\,|w|$. The argument satisfies $\arg(zw) = \arg(z) + \arg(w)$. These are direct consequences of the polar representation $z = r\exp(i\phi)$.

GEOMETRIC INTERPRETATION. If $z = x + iy$ is identified as a vector $\begin{bmatrix} x \\ y \end{bmatrix}$, then multiplication with an other complex number w is a **dilation-rotation**: a scaling by $|w|$ and a rotation by $\arg(w)$.

THE DE MOIVRE FORMULA $z^n = \exp(in\phi) = \cos(n\phi) + i\sin(n\phi) = (\cos(\phi) + i\sin(\phi))^n$ follows directly from $z = \exp(i\phi)$ but it is magic: it leads for example to formulas like $\cos(3\phi) = \cos(\phi)^3 - 3\cos(\phi)\sin^2(\phi)$ which would be more difficult to come by using geometrical or power series arguments. This formula is useful for example in integration problems like $\int \cos(x)^3\, dx$.

THE UNIT CIRCLE. Complex numbers of length 1 have the form $z = \exp(i\phi)$ and are located on the **unit circle**. The characteristic polynomial $f_A(\lambda) = \lambda^5 - 1$ of the matrix $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$ has all roots on the unit circle.

THE LOGARITHM. $\log(z)$ is defined for $z \neq 0$ as $\log|z| + i\arg(z)$. For example, $\log(2i) = \log(2) + i\pi/2$. A classic riddle at this point is What is i^i? The logarithm is not defined at 0 and the imaginary part is define only up to 2π. For example, both $i\pi/2$ and $5i\pi/2$ are equal to $\log(i)$.

HISTORY. The struggle of what to do with $\sqrt{-1}$ is historically quite interesting. Nagging questions appeared for example when trying to find closed solutions of polynomials. Cardano (1501-1576) was one of the mathematicians who at least considered complex numbers but called them arithmetic subtleties which were "as refined as useless". With Bombelli (1526-1573), complex numbers found some practical use. Descartes (1596-1650) called roots of negative numbers "imaginary".

Although the Fundamental Theorem of Algebra was still not proved in the 18th century, and complex numbers were not fully understood, the square root of minus one $\sqrt{-1}$ was used more and more. Euler (1707-1783) made the observation $\exp(ix) = \cos x + i\sin x$ which has as a special case the **magic formula** $e^{i\pi} + 1 = 0$ which relat $0, 1, \pi, e$ in one equation.

For decades, many mathematicians still thought complex numbers were **a waste of time**. Others used complex numbers extensively in their work. In 1620, Girard suggested that an equation may have as many roots as its degree in 1620. Leibniz (1646-1716) spent quite a bit of time trying to apply the laws of algebra to complex numbers. He and Johann Bernoulli used imaginary numbers as integration aids. Lambert used complex numbers for map projections, d'Alembert used them in hydrodynamics, while Euler, D'Alembert, and Lagrange used them in their incorrect proofs of the fundamental theorem of algebra. Euler write first the symbol i for $\sqrt{-1}$.

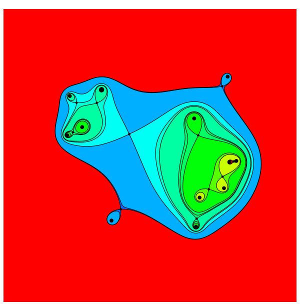
Gauss published the first correct proof of the fundamental theorem of algebra in his doctoral thesis, but still claimed in 1825 that **the true metaphysics of the square root of -1 is elusive** as late as 1825. By 1831 Gauss overcame his uncertainty about complex numbers and published his work on the geometric representation of complex numbers as points in the plane. In 1797, a Norwegian Caspar Wessel (1745-1818) and in 1806 a Swiss clerk named Jean Robert Argand (1768-1822) (who stated the theorem the first time for polynomials with complex coefficients) did similar work. But these efforts went unnoticed. William Rowan Hamilton (1805-1865) (who would also discover the quaternions while walking over a bridge) expressed in 1833 complex numbers as vectors.

Complex numbers continued to develop, for example into **complex function theory** or **chaos theory**. Complex numbers are helpful in geometry or number theory or quantum mechanics. Once believed impossible, ridiculous, and even fictitious they have become the most "natural numbers" (the natural numbers themselves are in fact the most "complex"). A philospher (who has taken Math21b!) who asks "does $\sqrt{-1}$ really exist?" might be shown the representation of $x + iy$ as $\begin{bmatrix} x & y \\ -y & x \end{bmatrix}$. When you add or multiply such dilation-rotation matrices, they behave like complex numbers: for example $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ plays the role of i.

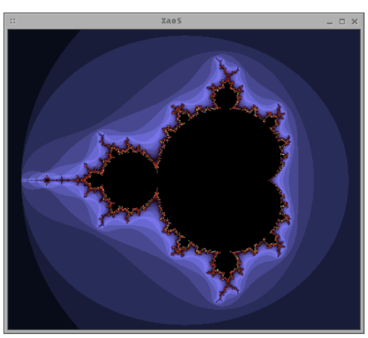
FUNDAMENTAL THEOREM OF ALGEBRA. (Gauss gave in 1799 the first proof) A polynomial of degree n has exactly n roots.

CONSEQUENCE: A $n \times n$ MATRIX HAS n EIGENVALUES. The characteristic polynomial $f_A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0$ satisfies $f_A(\lambda) = (\lambda - \lambda_n)\ldots(\lambda - \lambda_1)$, where λ_i are the roots of f.

TRACE AND DETERMINANT. Important facts:
- The trace of A is the sum of the eigenvalues: $\text{tr}(A) = \lambda_1 + \cdots + \lambda_n$. - The determinant of A is the product of the eigenvalues: $\det(A) = \lambda_1 \cdots \lambda_n$.



COMPLEX FUNCTIONS. The characteristic polynomial is an example of a function f from $\mathbf{C}$ to $\mathbf{C}$. The graph of this would live in a four dimensional space. One can visualize the function by drawing $z \mapsto |f(z)|$ which is a function from the complex plane to the reals. The figure to the left shows the contour lines of such a function if f is a polynomial.



ITERATION OF POLYNOMIALS. A topic which is off this course (it would be a course by itself) is the iteration of polynomials like $f_c(z) = z^2 + c$. The set of parameter values c for which the iterates $f_c(0), f_c^2(0) = f_c(f_c(0)), \ldots, f_c^n(0)$ stay bounded is called the **Mandelbrot set**. It is the fractal black region in the picture to the left. It appears everywhere, from photoshop plugins to decorations. In Mathematica, you can compute the set very quickly (see http://www.math.harvard.edu/computing/math/mandelbrot.m).

COMPLEX NUMBERS IN MATHEMATICA OR MAPLE. In both computer algebra systems, you use the letter I for $i = \sqrt{-1}$. In Maple for example, you can ask $\log(1 + I)$, in Mathematica, this would be Log[$1 + I$]. Eigenvalues or eigenvectors of a matrix will in general involve complex numbers. For example, in Mathematica, Eigenvalues[A] gives the eigenvalues of a matrix A and Eigensystem[A] gives the eigenvalues and the corresponding eigenvectors.

EIGENVALUES AND EIGENVECTORS OF A ROTATION. The rotation matrix $A = \begin{bmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{bmatrix}$ has the characteristic polynomial $\lambda^2 - 2\cos(\phi) + 1$. The eigenvalues are $\cos(\phi) \pm \sqrt{\cos^2(\phi) - 1} = \cos(\phi) \pm i\sin(\phi) = \exp(\pm i\phi)$. The eigenvector to $\lambda_1 = \exp(i\phi)$ is $v_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$ and the eigenvector to the eigenvector $\lambda_2 = \exp(-i\phi)$ is $v_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$.