Lecture 28

Maxima and Minima

Reminders

2. Midterm 11/10/2022: 5:30-7:30
Focus on 15-25 Psets. So, today's lecture is the last topic. So, CURVES, SURFACES, Line and surface INTEGRALS, Linear and Quadratic APPROXIMATION, GREEN and FTOLI are the main topics.

Putnam

Saturday, December 3, 2022.
Session A is scheduled from
10:00 - 1:00 p.m. and
Session B is scheduled from
3:00 p.m. - 6:00 p.m.
in Science Center Hall C

The 82nd William Lowell Putnam Mathematical Competition 2021

A1 A grasshopper starts at the origin in the coordinate plane and makes a sequence of hops. Each hop has length 5, and after each hop the grasshopper is at a point whose coordinates are both integers; thus, there are 12 possible locations for the grasshopper after the first hop. What is the smallest number of hops needed for the grasshopper to reach the point $(2021, 2021)$?

A2 For every positive real number x, let

$$g(x) = \lim_{r \to 0} \left((x+1)^{r+1} - x^{r+1} \right)^{\frac{1}{r}} .$$

Find $\lim_{x \to \infty} \frac{g(x)}{x}$.

A3 Determine all positive integers N for which the sphere

$$x^2 + y^2 + z^2 = N$$

has an inscribed regular tetrahedron whose vertices have integer coordinates.

A4 Let

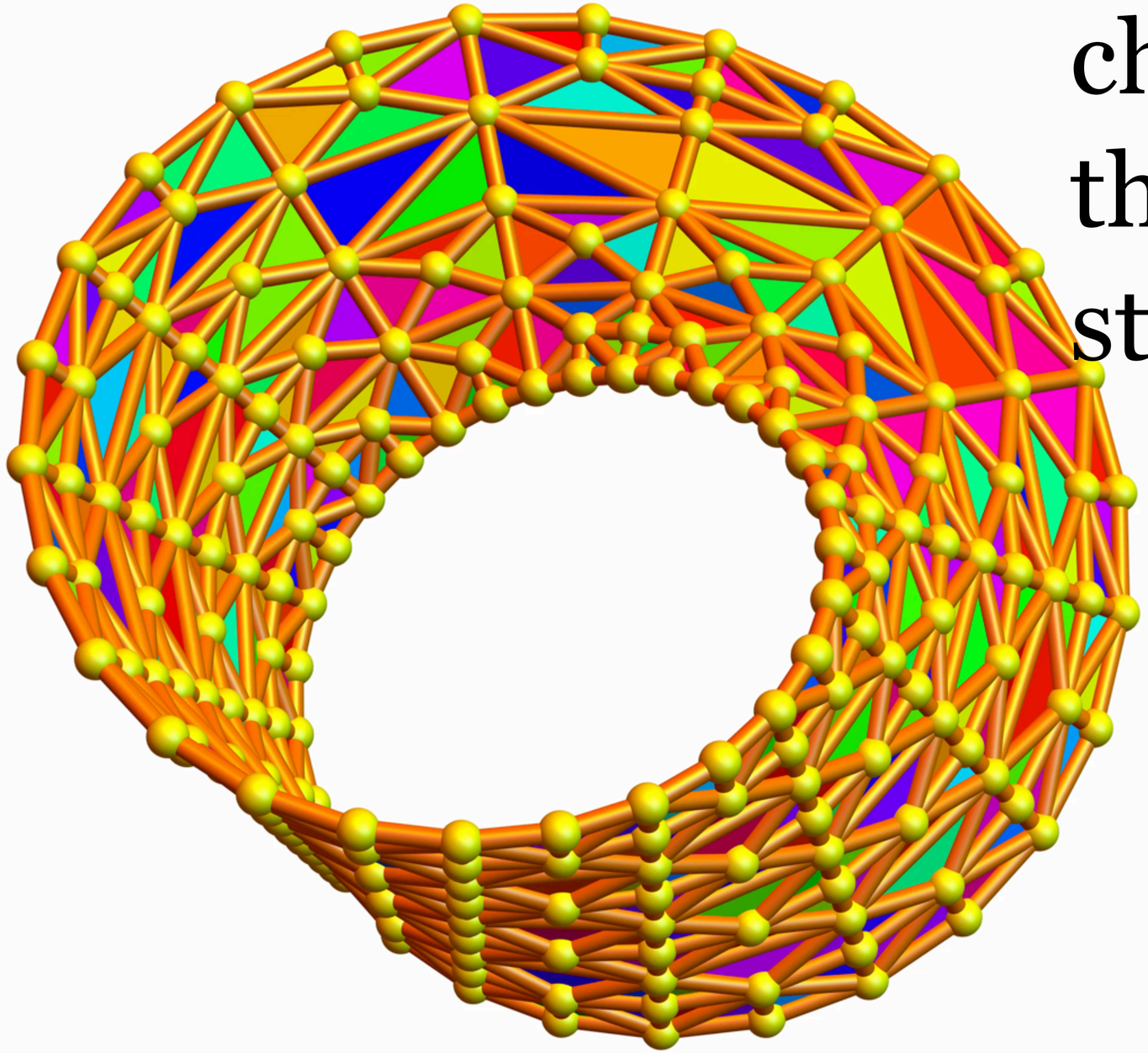
$$I(R) = \iint_{x^2+y^2 \leq R^2} \left(\frac{1+2x^2}{1+x^4+6x^2y^2+y^4} - \frac{1+y^2}{2+x^4+y^4} \right) dx\, dy\,.$$

Find

$$\lim_{R \to \infty} I(R)\,,$$

or show that this limit does not exist.

Why did the chicken cross the Moebius strip?



To get to the same side!

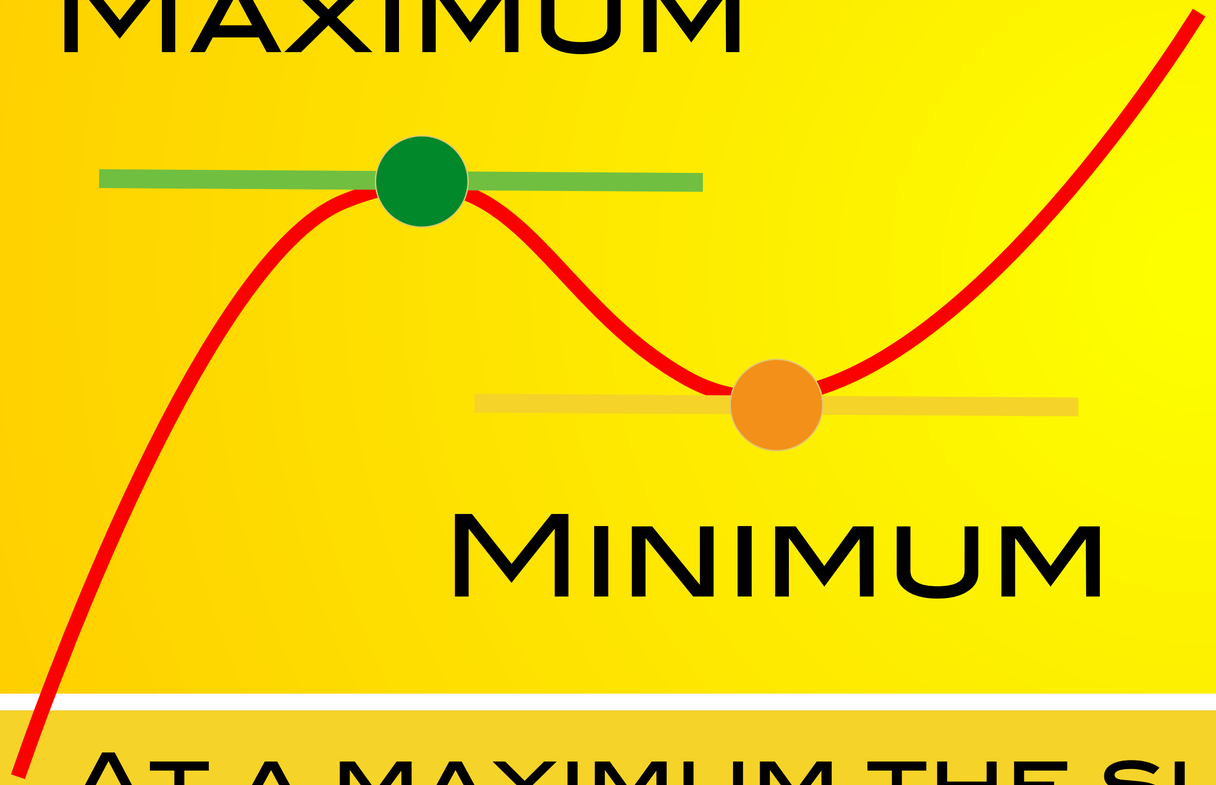
Nice to compute the flux !

Table of Contents

One Dimensional Review

Fermat

Maximum



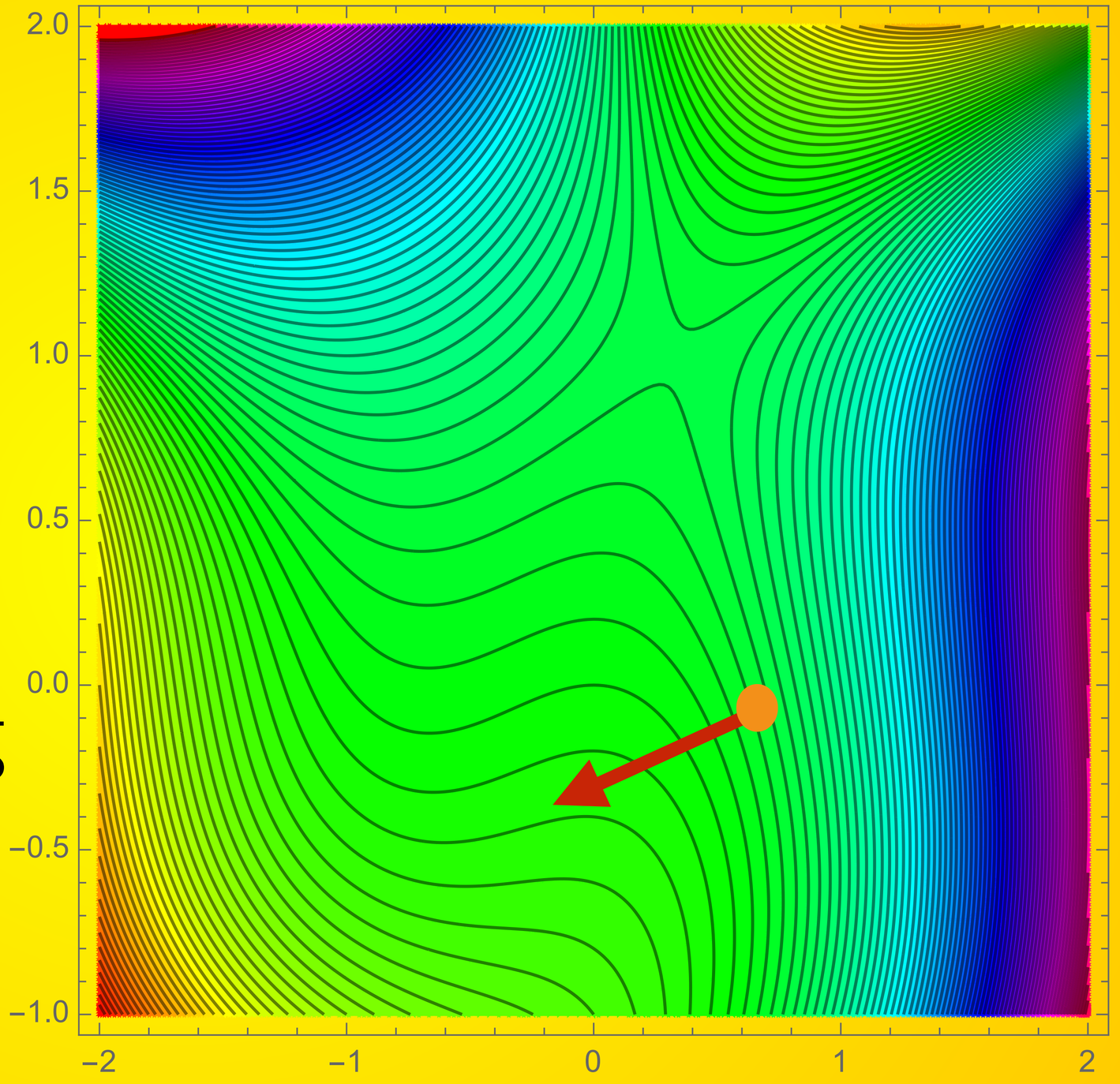
Minimum

At a maximum the slope is zero

1607-1665

Fermat:

If x is not a critical point, we can increase f by moving in the direction of the gradient



Reason

$$\frac{d}{dt}(f(r(t)) = \nabla f(r(t)) \cdot r'(t)$$

Apply the chain rule to

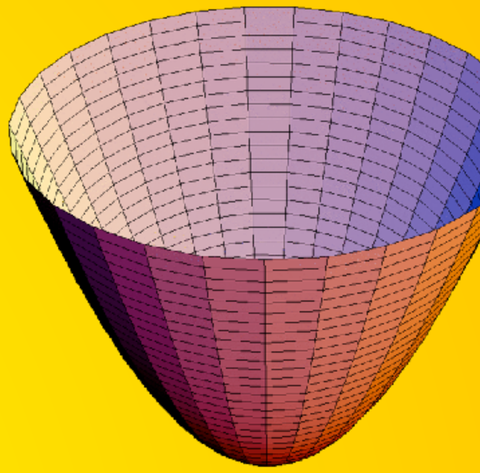
$$r(t) = x + t\nabla f(x)$$

to see it is positive if $\nabla f(x)$ is not zero.

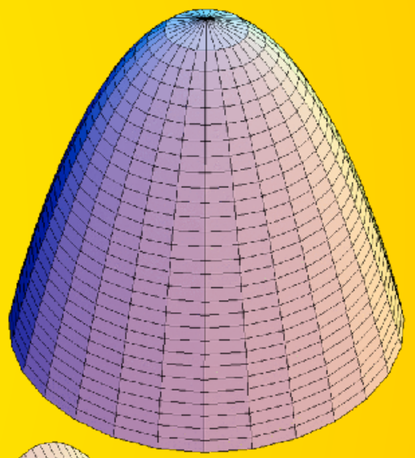

Mt.
Ephraim
Reservoirs
Turkey
Hill
ARLINGTON
ARLINGTON
Arlington
Heights
Arlington
Brook
BOSTON & MAINE

Second Derivative Test

Second derivative test

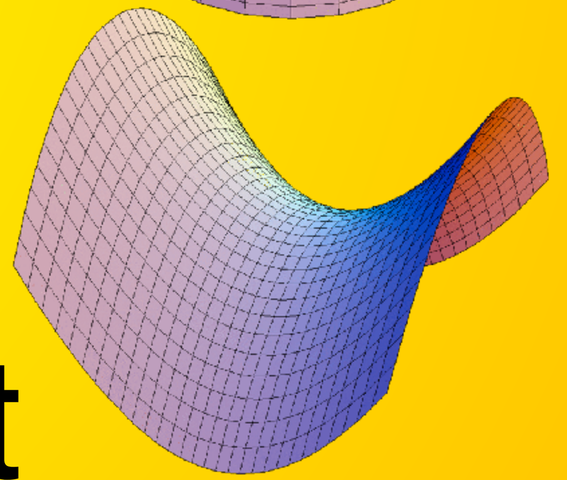


$D > 0 \quad f_{xx} > 0$ min



$D > 0 \quad f_{xx} < 0$ max

$D < 0$ saddle



$D = f_{xx}f_{yy} - f_{xy}^2$ discriminant

Proof

$$Z = f(x_0, y_0) \qquad B = f_{xy}(x_0, y_0)$$

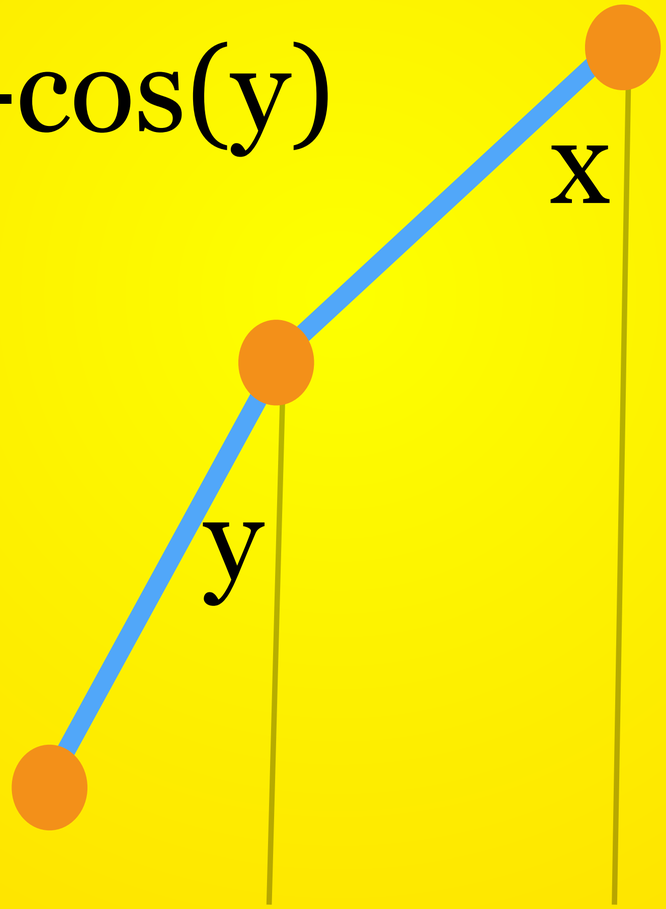
$$A = f_{xx}(x_0, y_0) \qquad C = f_{yy}(x_0, y_0)$$

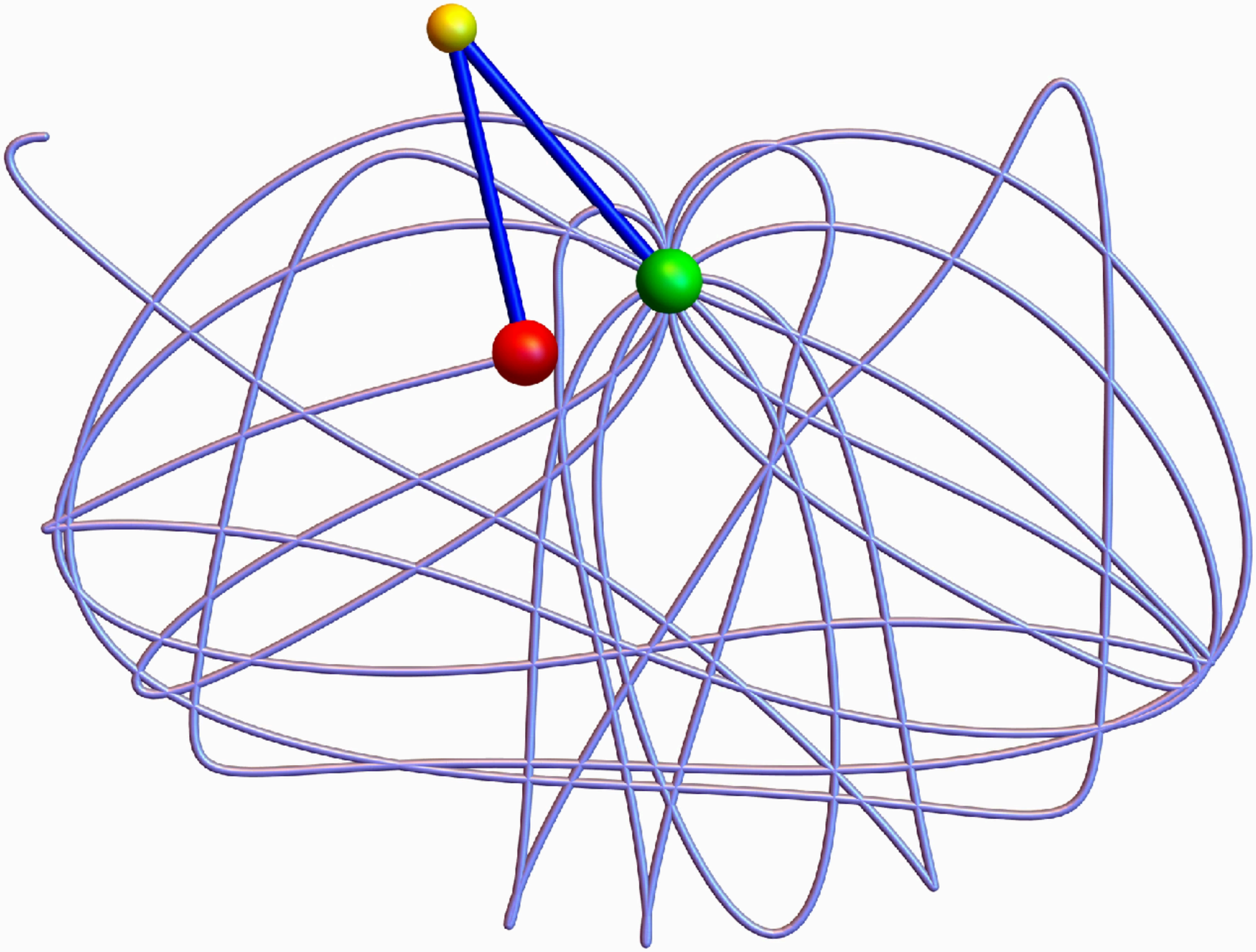
$$f = Z + Au^2 + 2Buv + Cv^2$$

Example

Double Pendulum

$$f(x,y) = -\cos(x)-\cos(y)$$





Code

```
=  x^3 - 2 y^2 - 12 x*y;
ClassifyCriticalPoints[f_, {x_, y_}] :=
 Module[{X, P, H, g, d, S}, X = {x, y};
  P = Sort[Solve[Thread[D[f, #] f & /@ X == 0], X]];
  H = Outer[D[f, #1, #2] &, X, X]; g = H[[1, 1]]; d = Det[H];
  S[d_, g_] := If[d < 0, "saddle", If[g > 0, "minimum", "maximum"]];
  TableForm[{x, y, d, g, S[d, g], f} /. P,
   TableHeadings -> {None, {x, y, "D", "f_xx", "Type", "f"}}]]
ClassifyCriticalPoints[f, {x, y}]
```

THE END