MULTIVARIABLE CALCULUS

OLIVER KNILL, MATH 21A

Lecture 25: Fundamental Theorem of Line Integrals

The theorem

The fundamental theorem of line integrals is:

If $\vec{F} = \nabla f$ is a gradient field then $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt = f(\vec{r}(b)) - f(\vec{r}(a))$.

Proof: chain rule: $\int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt = \int_a^b \frac{d}{dt} f(\vec{r}(t))\; dt = f(\vec{r}(b)) - f(\vec{r}(a))$. The last step is due to the **fundamental theorem of calculus**.

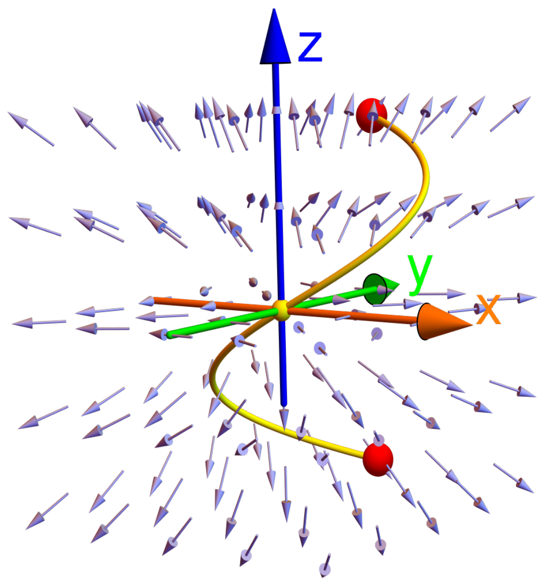

FIGURE 1. We see the gradient field $\vec{F}(x, y, z) = \langle y, x, z \rangle$. It has the potential $f(x, y, z) = xy + z^2/2$. Instead of computing the line integral along the path $\vec{r}(t)$ from A to B, we can just evaluate the function at A and B and form $f(B) - f(A)$.

Remarks

1) We understand now conservative fields better. If $\vec{F}$ is a gradient field, then we have the closed loop property: the line integral along any closed loop is zero.

2) The theorem can also be formulated as follows. If C is a curve and δC is the boundary (consisting of two points A, B), then $\int_C \nabla f \; d\vec{r} = f(B) - f(A)$. This also works for the fundamental theorem of calculus: $\int_A^B f'(t) \; dt = f(B) - f(A)$.

3) Let us look again at Green's theorem

$$\boxed{\int_C \vec{F}(\vec{r}(t)) \cdot \; \vec{r}\,'(t) \; dt = \iint_G \text{curl}(\vec{F}) \; dA.}$$

What happens if $\vec{F}$ is a gradient field? The left hand side is zero because of the fundamental theorem of line integrals and because a closed curve C has no boundary! The right hand side is zero because the curl of any gradient field is zero everywhere.

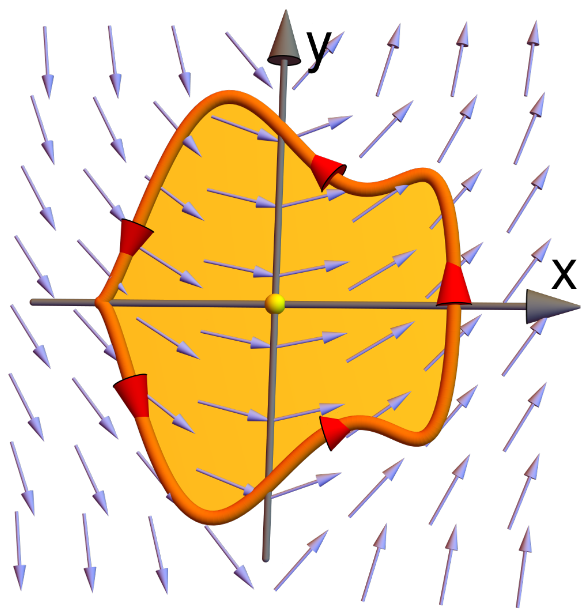

FIGURE 2. If $\vec{F}$ is a gradient field, then Green's theorem can be understood more easily. The boundary line integral is zero because of the fundamental theorem of line integrals. The double integral is zero because the curl is zero for a gradient field.

4) Note that orientation matters like for the fundamental theorem of calculus. If we pass the curve in the opposite direction, then the sign changes. So, unlike arc length $\int_a^b |\vec{r}\;'(t)| \; dt$, for which the orientation does not matter and which is always non-negative, the line integral is **orientation sensitive**.

5) GREEN and the FTLI already give a complete list of integral theorems in the plane. In 3D, there will be three theorems. The FTLI, STOKES and GAUSS. Reaching these theorems is the ultimate goal of this course.

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