MULTIVARIABLE CALCULUS

OLIVER KNILL, MATH 21A

Lecture 20: Line integrals

Line integrals

If $\vec{r}(t) = \langle x(t), y(t) \rangle$ is a planar curve C for $t \in [a, b]$ and $\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ a vector field. Define the **line integral**

$$\int_C \vec{F}\ d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt\ .$$

Think of $\vec{F}$ as a force and $\vec{r}\,'$ as velocity so that $P(t) = \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)$ is **power** and $\int_a^b P(t)\ dt$ is **work**. The sign is now important. If a curve is traced backwards, the line integral changes sign.

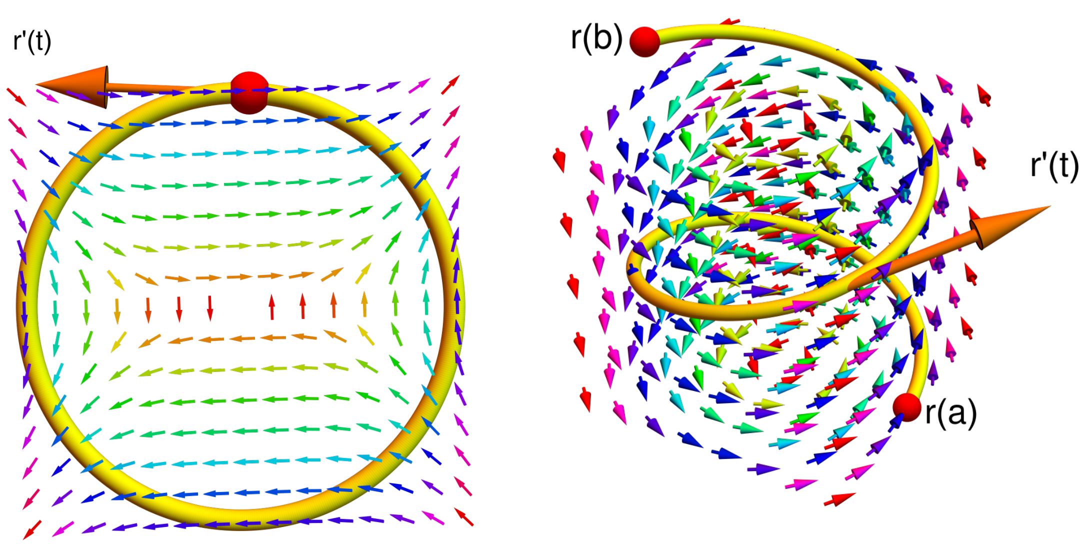

FIGURE 1. Picture of a line integral of a planar vector field and a line integral in space.

Example

Let $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$ with $0 \le t \le 2\pi$ be a circle and $\vec{F} = \langle y, x^2 \rangle$ be a vector field. First find $\vec{F}(\vec{r}(t)) = \langle \sin(t), \cos(t)^2 \rangle$, then the velocity $\vec{r}\,'(t) = \langle -\sin(t), \cos(t) \rangle$ so that the power is $P(t) = \langle \sin(t), \cos(t)^2 \rangle \cdot \langle -\sin(t), \cos(t) \rangle = -\sin^2(t) + \cos^3(t)$. Integrating this from 0 to 2π gives $-\pi$.

Perpetual motion

If we integrate along a closed loop and get a positive line integral, we gained energy. We could repeat that and have a source of energy.

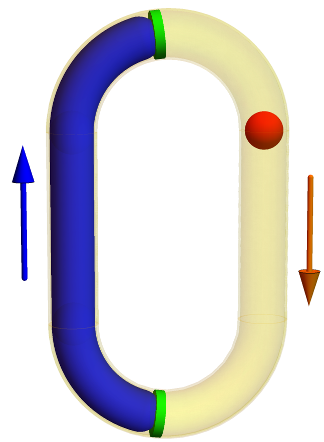

Figure 2. Can we realize without using any energy a vector field in space for which the line integral along some closed loop is non-zero?

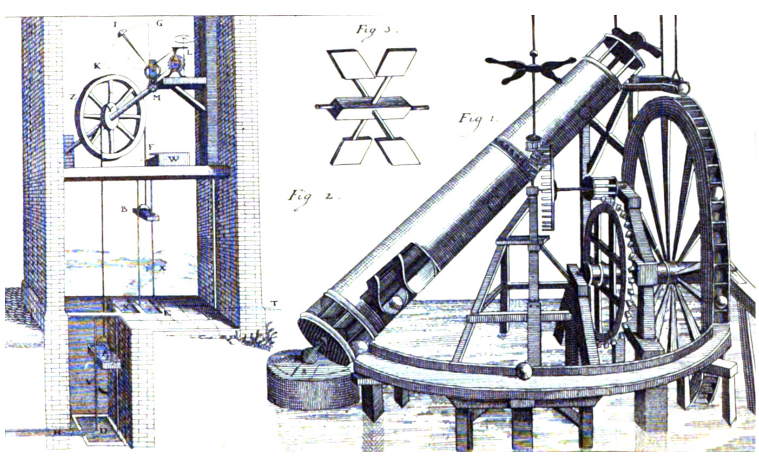

Figure 3. This has been a dream for hundreds of years

Outlook

We will see Monday the connection between curl and line integrals. But we already can see that if a vector field has positive curl, the line integral of a small circle traced **counter clockwise** is positive. Somehow, we we will be able to link curl and line integrals. This will be explained in the next lecture.

Section Oliver Knill, knill@math.harvard.edu, Harvard College, Fall 2022