MULTIVARIABLE CALCULUS

OLIVER KNILL, MATH 21A

Lecture 15: Curves

Parametric curves

A vector valued function

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

is called a **parametrized curve**. If the curve is in the plane, we write

$$\vec{r}(t) = \langle x(t), y(t) \rangle$$

and talk about a **planar curve**. We distinguish the **parametrization** which is map from $\mathbb{R}$ to $\mathbb{R}^3$ or $\mathbb{R}$ to $\mathbb{R}^2$ or then the actual picture which the curve. It is a good idea to think of t as "time" and the image of the map as the **path** traced by an object. The parametrization does not only give the path, it describes also **how we trace the path**. For example, it allows us to get the **velocity** $\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle$. The magnitude of the velocity is called the **speed**.

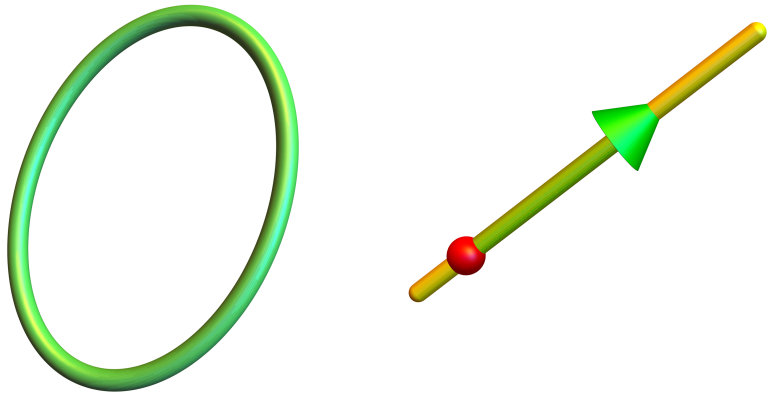

FIGURE 1. A circle $\vec{r}(t) = \langle \cos(t), \sin(t), \sin(t) \rangle$ and a line $\vec{r}(t) = \langle 1, 2, 3 \rangle + t\langle 1, 1, 1 \rangle$ given by a point $P = (1, 2, 3)$ and vector $\vec{v} = \langle 1, 1, 1 \rangle$.

Here is an ellipse in space obtained by intersecting two cylinders $x^2+y^2 = 1, x^2+z^2 = 1$:

$$\vec{r}(t) = \langle \cos(t), \sin(t), \sin(t) \rangle \ .$$

If $t \in [0, 2\pi)$ we trace every point of the curve exactly once. The curve

$$\vec{r}(t) = \langle \cos(t + \sin(t)), \sin(t + \sin(t)), \sin(t + \sin(t)) \rangle \ .$$

traces the same ellipse, but now the curve is traced in a different way.

Here is a **helix**

$$\vec{r}(t) = \langle \cos(5t), \sin(5t), t \rangle$$

and a curve

$$\vec{r}(t) = \langle 2\cos(t), 3\sin(t), 6\cos(t)\sin(t) \rangle$$

The picture shows both of them.

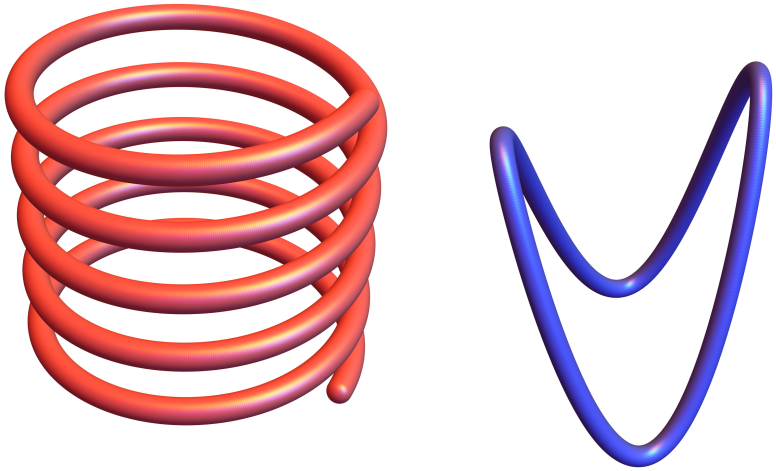

FIGURE 2. An example of a helix. The second curve is the intersection of an elliptical cylinder $x^2/4 + y^2/9 = 1$ with a hyperbolic paraboloid $z = xy$.

An example of a knot is the so called **torus knot**:

$$\vec{r}(t) = \langle (3 + \cos(5t))\cos(3t), (3 + \cos(5t))\sin(3t), 2\sin(5t) \rangle \ .$$

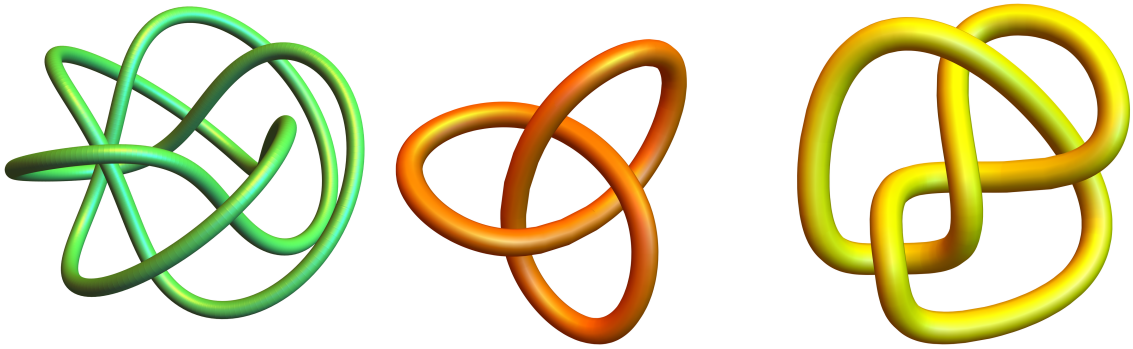

FIGURE 3. A torus knot, the trefoil and the figure eight knot