MULTIVARIABLE CALCULUS

OLIVER KNILL, MATH 21A

Lecture 14: Problem solving

Distance problems

How can we find the distance between two objects A, B? We will discuss in class some problems like

- If A is a point and B is a point.
- If A is a point and B is a plane.
- If A is a point and B is a line.
- If A is a line and B is a line.
- If A is a sphere and B is a cylinder.
- If A is a cylinder and B is a cylinder.

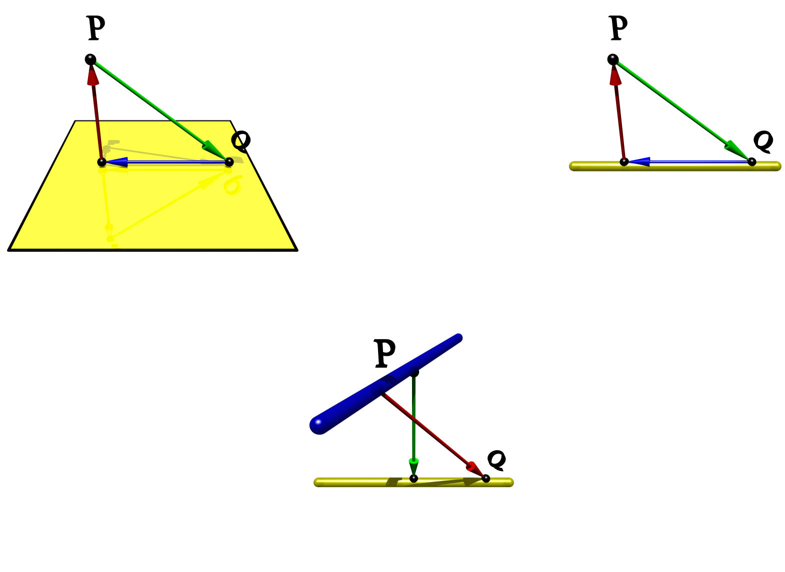


Figure 1. Distance between point and plane, a line and a point and distance between two lines.

In order to solve these problems, we need both **tools** and **ideas**. An idea when used several times then becomes a **method**.

Tools

For distance formulas we have great tools already for computing distances. We will discuss some of this in class:

- Dot product and Scalar component.
- Cross product and Area.
- Volume and area.
- Using calculus.

Then we need to make these tools work. For that we need **problem solving techniques**.

Problem solving strategies

There are many books and online resources about problem solving. Interest has amplified because we also work on machines gain such capabilities. Whether this is a good idea or not is an other question.

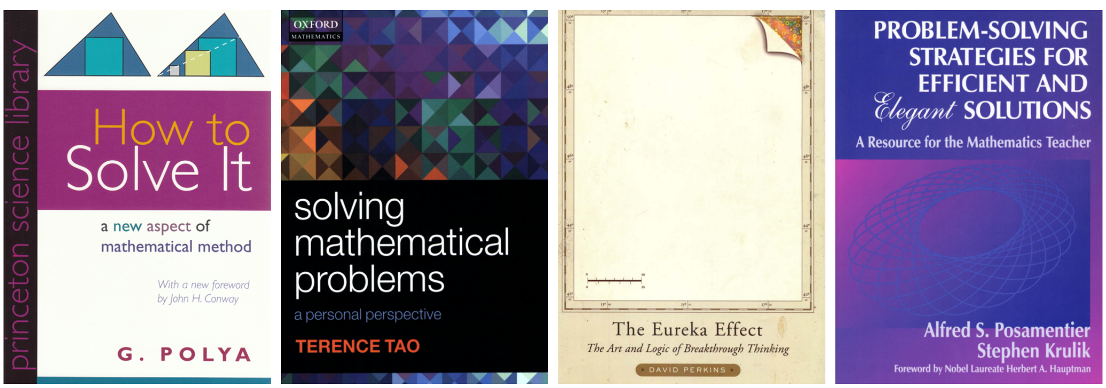


Figure 2. 4 Superstar books: Polya: How to Solve it. Tao: Solving Mathematical Problems, Perkins: The Eureka effect, Posamentier-Krulik: Problem solving strategies.

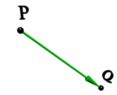
Let me just add personal note: I myself became a fan of Polya's book as a first year undergraduate student. It immediately helped me to become a better student. By the way, Polya's "How to solve it" was published in 1945. His now famous **list** is:

Polya principles
1. **Understand** the problem: unknowns, data, draw figure.
2. Devise a **plan**: similar or related problem?
3. **Carry out** the plan: check each step.
4. **Examine** the solution: can other problems be solved as such?

We practice this here in the context of distances. The following attached pages were written in 2002 for Math 21a and have undergone very little change over the last 20 years. The graphics was done using the ray tracer Povray. I personally am a big fan of distance formulas because if you understand them you understand everything you need about vectors: dot product, cross product, projection, area and volume.

DISTANCE POINT-POINT. The distance between P and Q is

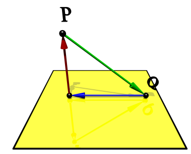
$$d(P,Q) = |\vec{PQ}| \; .$$



DISTANCE POINT-PLANE. The distance between a point P and a plane Σ given by $ax+by+cz = \vec{n}\cdot\vec{x} = d$ containing Q is

$$d(P,\Sigma) = \frac{|\vec{PQ}\cdot\vec{n}|}{|\vec{n}|}$$

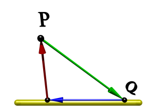
See it as the scalar projection of $\vec{PQ}$ onto $\vec{n} = [a,b,c]$. If the plane is parametrized $\vec{r} = \vec{OQ} + t\vec{v} + s\vec{w}$, then, with $\vec{n} = \vec{v}\times\vec{w}$, we can use that "height=volume/base area" holds in a parallelepiped.



DISTANCE POINT-LINE . If P is a point in space and L is the line $\vec{r}(t) = 0\vec{Q} + t\vec{u}$, then

$$d(P,L) = \frac{|\vec{PQ}\times\vec{u}|}{|\vec{u}|}$$

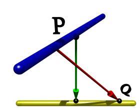
is the distance between P and the line L. Proof: the area divided by base length is the height of the parallelogram spanned by $\vec{PQ}$ and $\vec{u}$.



DISTANCE LINE-LINE . L is the line $\vec{r}(t) = Q + t\vec{u}$ and M is the line $\vec{s}(t) = P + t\vec{v}$, then

$$d(L,M) = \frac{|\vec{PQ}\cdot(\vec{u}\times\vec{v})|}{|\vec{u}\times\vec{v}|}$$

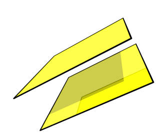
is the distance between the two lines L and M. Proof: the distance is the length of the vector projection of $\vec{PQ}$ onto $\vec{u}\times\vec{v}$.



DISTANCE PLANE-PLANE . If $\vec{n}\cdot\vec{x} = d$ and $\vec{n}\cdot\vec{x} = e$ are two parallel planes, then their distance is

$$\frac{|e-d|}{|\vec{n}|} \; .$$

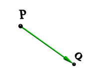
Proof. If $\vec{x}$ satisfies $\vec{n}\cdot\vec{x} = d$, then $\vec{y} = \vec{x} + \vec{u}$ satisfies $\vec{n}\cdot\vec{y} = e$, where $\vec{u} = (e-d)\vec{n}/|\vec{n}|^2$ is perpendicular to the planes and has length $(e-d)/|\vec{n}|$. Non-parallel planes have distance 0.



DISTANCE POINT-POINT . $P = (-5,2,4)$ and $Q = (-2,2,0)$ are two points, then

$$d(P,Q) = |\vec{PQ}| = \sqrt{(-5+2)^2 + (2-2)^2 + (0-4)^2} = 5$$

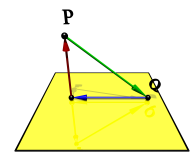
A question: what is the distance between the point $(-5,2,4)$ and the sphere $(x+2)^2 + (y-2)^2 + z^2 = 1$?



DISTANCE POINT-PLANE . $P = (7,1,4)$ is a point and $\Sigma : 2x + 4y + 5z = 9$ is a plane which contains the point $Q = (0,1,1)$. Then

$$d(P,\Sigma) = \frac{|[-7,0,-3] \cdot [2,4,5]|}{|[2,4,5]|} = \frac{29}{\sqrt{45}}$$

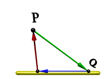
is the distance between P and Σ. Something to think about: without the absolute value, the result could become negative. What does this tell about the point P?



DISTANCE POINT-LINE . $P = (2,3,1)$ is a point in space and L is the line $\vec{r}(t) = [1,1,2] + t[5,0,1]$. Then

$$d(P,L) = \frac{|[-1,-2,1] \times [5,0,1]|}{[5,0,1]} = \frac{|[-2,6,10]|}{\sqrt{26}} = \frac{\sqrt{140}}{\sqrt{26}}$$

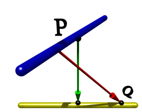
is the distance between P and L. Something to think about: what is the equation of the plane which contains the point P and the line L?



DISTANCE LINE-LINE . L is the line $\vec{r}(t) = [2,1,4] + t[-1,1,0]$ and M is the line $\vec{s}(t) = [-1,0,2] + t[5,1,2]$. The cross product of $[-1,1,0]$ and $[5,1,2]$ is $[2,2,-6]$. The distance between these two lines is

$$d(L,M) = \frac{|[3,1,2] \cdot [2,2,-6]|}{|[2,2,-6]|} = \frac{4}{\sqrt{44}} .$$

Something to think about: also here, without the absolute value, the formula can give a negative result. What does this mean geometrically?



DISTANCE PLANE-PLANE . $5x+4y+3z = 8$ and $5x+4y+3z = 1$ are two parallel planes. Their distance is

$$\frac{|8-1|}{|[5,4,3]|} = \frac{7}{\sqrt{50}}$$

Something to think about: what is the distance between the planes $x + 3y - 2z = 2$ and $5x + 15y - 10z = 30$?