OLIVER KNILL, MATH 21A

Lecture 11: Vectors and Lines

Vectors

Two points $P = (a, b, c)$ and $Q = (x, y, z)$ in space $\mathbf{R}^3$ define a **vector** $\vec{v} = \begin{bmatrix} x - a \\ y - b \\ z - c \end{bmatrix}$.

We can draw vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$. The magnitude of a vector $\vec{v} = \langle a, b, c\rangle$ is $\|\vec{v}\| = \sqrt{a^2 + b^2 + c^2}$. The direction of a **non-zero vector** is $\vec{v}/\|\vec{v}\|$. Note that the zero vector has no direction so that the definition given in many textbooks is incorrect. The zero vector is an important vector too but it has no direction. Also, a definition should define a quantity. A movie has a direction and a magnitude, but we are not talking here about movies!

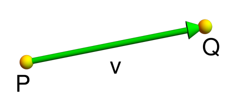

Figure 1. A vector $\vec{PQ}$ is defined by two points P and Q. The components of a vector are the difference between the coordinates. If $P = (2, 2, 5)$ and $Q = (5, 8, 2)$, then $\vec{PQ} = \langle 3, 6, -3\rangle$. Think of what you have to add to P to get to Q.

Operations

Vectors can be **added**, **subtracted** or **multiplied with a scalar**. Algebraically this is done by adding the components:

$$\langle 3, 4, 5\rangle + \langle 1, 2, 1\rangle = \langle 4, 6, 6\rangle \ .$$
$$\langle 3, 4, 5\rangle - \langle 1, 2, 1\rangle = \langle 2, 3, 4\rangle \ .$$
$$(-2)\langle 3, 4, 5\rangle = \langle -6, -8, -10\rangle \ .$$

Geometrically, adding the vectors means getting the main diagonal of the parallelogram spanned by the two vectors. The difference is in the side diagonal of the parallelogram. The scalar multiple just scales a vector. It produces a vector which is **parallel** to the original vector. Note that we also can multiply with 0. The result is then the vector $\vec{0}$.

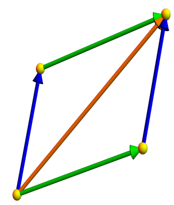

FIGURE 2. The addition of two vectors produces a vector in which one goes first along the first vector and the along the second vector. Doing this in the two possible orders produces a parallelogram.

SCALAR COMPONENT

The **scalar component** of a vector $\vec{b}$ in the direction of $\vec{a}$ is called $\text{comp}_{\vec{a}}\vec{b}$. A formula will be given in the next lecture. For $\vec{a} = \langle 1, 0, 0 \rangle$ it is the first component of the vector.

$$\text{comp}_{\langle 1,0,0 \rangle} \langle 3, 4, 5 \rangle = 3$$
$$\text{comp}_{\langle 0,1,0 \rangle} \langle 3, 4, 5 \rangle = 4$$
$$\text{comp}_{\langle 0,0,1 \rangle} \langle 3, 4, 5 \rangle = 5$$

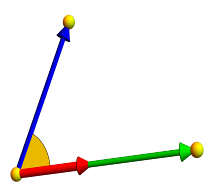

FIGURE 3. The component $\text{comp}_{\vec{a}}\vec{b}$ will be defined as the scalar $\vec{a} \cdot \vec{b}/\vec{b}$. It is positive if the angle between the two vectors is smaller than $\pi/2$.

LINE

A point like $P = (1, 3, 4)$ and a non-zero vector $\vec{v} = \langle 5, 6, 7 \rangle$ define a line $\vec{r}(t) = \langle 1, 3, 4 \rangle + t\langle 5, 6, 7 \rangle$. This can also be written as $\vec{r}(t) = \langle x, y, z \rangle = \langle 1 + 5t, 3 + 6t, 4 + 7t \rangle$. If you like, you can also leave it as components like $x = 1 + 5t, y = 3 + 6t, z = 4 + 7t$.

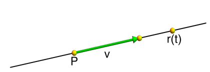

FIGURE 4. A line is defined by a point P and a vector $\vec{v}$.

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