Homework 31: Divergence Theorem

This last homework is due Monday, December 2, 2019. This is the last lecture.

1 Find the flux of the field $\vec{F}(x, y, z) = [z^3 + x^3, y^3 - 3yz^2, z^3 - \sin(x^3)]$ through the boundary of the solid bounded by paraboloid $z = 16 - x^2 - y^2$ and the xy-plane.

Solution:
$\text{div}\vec{F} = 3(x^2 + y^2)$. Use cylindrical coordinates to compute

$$\iiint_E \text{div}(\vec{F}) dV = \int_0^{2\pi} \int_0^4 \int_0^{16-r^2} 3r^2) r dz dr d\theta \ .$$

which simplifies to 2048π.

2 Find the flux of the vector field $\vec{F}(x, y, z) = [x^2y + \cos^6(y), xy^2, 2xyz + e^{\sin(x)}]$ through the outwards oriented solid bound by $x = 0, y = 0, z = 0$, and $x + 2y + z = 2$.

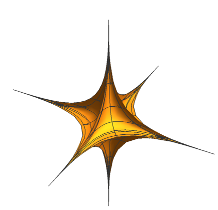
Solution:
The divergence of $\vec{F}$ is

$$\text{div}\vec{F} = \frac{\partial}{\partial x}(x^2y)+\frac{\partial}{\partial y}(xy^2)+\frac{\partial}{\partial z}(2xyz) = 2xy+2xy+2xy = 6xy.$$

Divergence Theorem turns the flux integral into a triple integral. Thus we get

$$\begin{aligned}
\iint_S \vec{F}\cdot d\vec{S} &= \iiint_E 6xy\,dV = \int_0^1\int_0^{2-2y}\int_0^{2-x-2y} 6xy\,dz\,dx\,dy \\
&= \int_0^1\int_0^{2-2y} 6xy(2-x-2y)dx\,dy \\
&= \int_0^1\int_0^{2-2y}(12xy-6xy^2-12xy^2)\,dx\,dy \\
&= \int_0^1\left[6x^2y-2x^3y-6x^2y^2\right]_{x=0}^{2-2y}\,dy \\
&= \int_0^1 y(2-2y)^3\,dy = \left[-\frac{8}{5}y^5+6y^4-8y^3+4y^2\right]_0^1 \\
&= \frac{2}{5}.
\end{aligned}$$

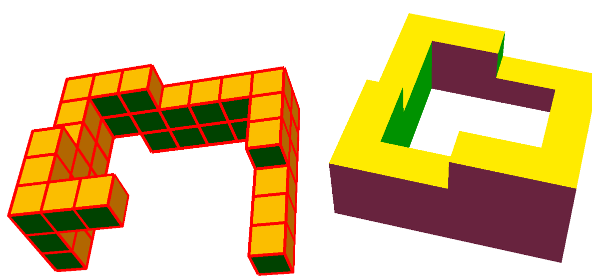
3 Compute the volume of the solid enclosed by the surface parametrized by $\vec{r}(u,v) = [\sin^5(v)\cos^5(u), \sin^5(v)\sin^5(u), \cos^5(v)]$ with $0 \leq u \leq 2\pi$ and $0 \leq v \leq \pi$. Use $F = [0,0,z]$ in order not to have to compute too much. You then only need to compute the last component of $r_u \times r_v$! You can then use Mathematica to compute the integral. If you get a negative sign, you might have to blame the orientation.



Solution:

Use $F = [0, 0, z]$ and find the flux of this vector field through the boundary. The computation of $r_u \times r_v$ is a bit complicated $[-25\sin^4(u)\cos(u)\sin^6(v)\cos^4(v), -25\sin(u)\cos^4(u)\sin^6(v)\cos^4(v), -2$ The last component times $\vec{F}(r(u,v) = [0, 0, \cos^5(u)]$ is $-25\sin^4(u)\cos^4(u)\sin^9(v)\cos^6(v)$. This gives $-(20\pi/3003)$ but because the orientation of the parametrization is outwards, we $(20\pi/3003)$.

4 Find $\iint_S \vec{F} \cdot d\vec{S}$, where $\vec{F}(x,y,z) = [-x + \sin(y^5) + \cos(e^z), 55y + \sin(z) + \sin(z), z + \sin(x) + e^y]$ and S is the boundary of the Escher stair solid displayed in the picture. The right picture shows the same figure from an other angle leading to the illusion. Each brick is a cube of unit length 1.



Solution:

a) By the Divergence Theorem, $\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} dV = 55$ (volume of E) $= 55 * 37$.

5 a) Use one of the integral theorems to evaluate $\int\int_S \operatorname{curl}(\vec{F}) \cdot d\vec{S}$, where $\vec{F}(x,y,z) = [x^2yz, yz^2, z^3e^{xy}]$, where is the part of upwards oriented surface $x^2 + y^2 + z^2 = 5$ that lies above the plane

$z = 1$.

b) Use one of the integral theorems to compute the line integral of $\vec{F}(x,y,z) = [x^3, y^5, 2z]$ along the path $\vec{r}(t) = [\cos(t) + t^{21}\sin(17t), \sin(t) + \sin(20t), t]$ from $t = 0$ to $t = 10\pi$.

Solution:

a) $\iint_S \text{curl}\vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r}$ where C:

$$\begin{aligned} \vec{r}(t) &= [2\cos t, 2\sin t, 1], 0 \le t \le 2\pi \\ \vec{r}\,'(t) &= [-2\sin t, 2\cos t, 0] \\ \vec{F}(\vec{r}(t)) &= [8\cos^2 t \sin t, 2\sin t, e^{4\cos t \sin t}] \\ \text{and } \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) &= -16\cos^2 t \sin^2 t + 4\sin t \cos t \end{aligned}$$

Thus $\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (-16\cos^2 t \sin^2 t + 4\sin t \cos t)dt = \left[-16\left(-\frac{1}{4}\sin t \cos^3 t + \frac{1}{16}\sin 2t + \frac{1}{8}t\right) + 2\sin^2 t\right]_0^{2\pi} = -4\pi$

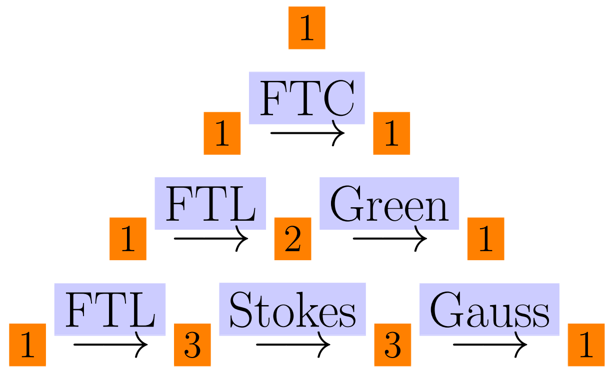
b) This is a gradient field with potential $x^4/4 + y^5/5 + z^2$. The initial point is $(1,0,0)$. The end point is $(1,0,10\pi)$. The result is $100\pi^2$.

Main points

Divergence Theorem. $\iiint_E \text{div}(\vec{F})\, dV = \iint_S \vec{F} \cdot d\vec{S}$. All integral theorems are incarnations of **the fundamental theorem of multivariable Calculus**

$$\int_G dF = \int_{\delta G} F$$

where dF is a **derivative** of F and δG is the **boundary** of G.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FTC
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Green
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FTL
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Stokes
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Gauss
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