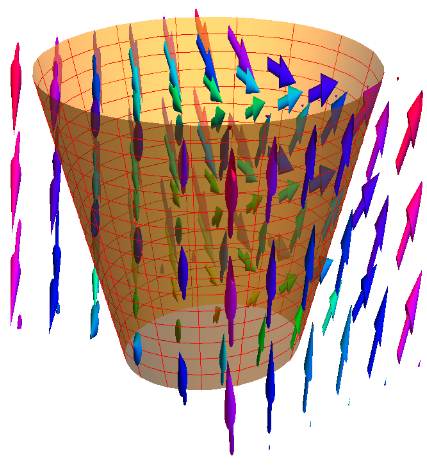
Homework 30: Stokes Theorem

This homework is due Monday, 11/25.

1 Evaluate $\int\int_S \text{curl}(\mathbf{F}) \cdot d\vec{S}$, where $\vec{F}(x,y,z) = [-7y, x, z + \sin(z-1)\sin(z-2)]$, and where S is the part of the cone $z^2 = x^2 + y^2$ that lies between the planes $z = 1$ and $z = 2$. The cone can be parametrized by $\vec{r}(u,v) = [v\cos(u), v\sin(u), v]$ with $0 \le u \le 2\pi$ and $1 \le v \le 2$.



2 Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x,y,z) = [x^3 - x^5, 7e^x, z^5 + z^9]$ and where C is the boundary of the part of the plane $6x+3y+z = 12$ in the first octant, oriented counterclockwise as viewed from above.

3 Evaluate the line integral $\int_C \vec{F} \cdot d\mathbf{r}$, where $\vec{F}(x,y,z) = [xy, 2z, 3y]$ and C is the curve of intersection of the plane $x + z = 5$ and the cylinder $x^2 + y^2 = 9$, oriented counterclockwise as viewed from above.

4 Compute both sides of Stokes' Theorem for $\vec{F}(x,y,z) = [-2yz, y, 3x]$ and the surface S which is the part of the paraboloid $z = 5 - x^2 - y^2$ that lies above the plane $z = 1$, oriented upwards.

5 a) Evaluate $\int_C \vec{F}(x,y,z) \cdot d\vec{r}$ with $\vec{F}(x,y,z) = [y + \sin x, z^2 + \cos y, x^3]$, where C is the curve $\vec{r}(t) = [\sin t, \cos t, \sin 2t], 0 \leq t \leq 2\pi$ which as you can see lies on the surface $z = 2xy$.
b) Explain without doing any computation that if S is the torus $\vec{r}(u,v) = [(2+\cos(v))\cos(u), (2+\cos(v))\sin(u), \sin(v)]$ with $0 \leq u \leq 2\pi, 0 \leq v \leq 2\pi$ and $\vec{F}$ is a vector field like $\vec{F}(x,y,z) = [e^{e^{e^x}}, \sin\sin(y+z+x), x^{100}]$ then $\int\int_S \text{curl}(\vec{F}) \cdot d\vec{S} = 0$.

Main points

Stokes theorem: Let S be a surface bounded by a curve C and $\vec{F}$ be a vector field. Then

$$\int\int_S \text{curl}(\vec{F}) \ \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} \ .$$

The orientation of S is given by the parametrization: the orientation of C is such that if you walk along C with the head in the "up" direction $\vec{r}_u \times \vec{r}_v$ and your nose into the $\vec{r}\,'$ direction, then your left foot is on the surface.

Written out in detail, we have

$$\int\int_R \text{curl}(\vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) \ dudv = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \ dt$$

A lot of things come together here: surfaces, curves, dot product, cross product, triple scalar product, vector fields, double integrals and curl. What does it mean? From a SMBC cartoon: **"Stokes theorem? Yeah, that's how if you draw a loop around something, you can tell how much swirly is in it."**