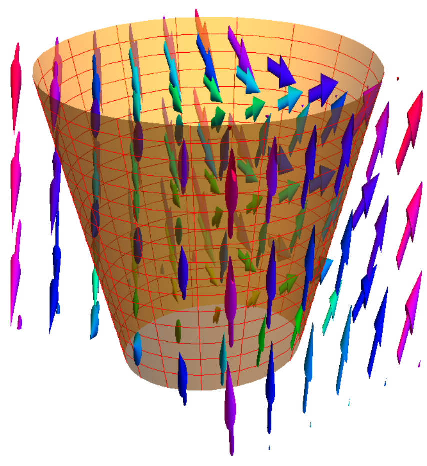
Homework 30: Stokes Theorem

This homework is due Monday, 11/25.

1 Evaluate $\int\int_S \mathrm{curl}(\mathbf{F}) \cdot d\vec{S}$, where $\vec{F}(x, y, z) = [-7y, x, z + \sin(z-1)\sin(z-2)]$, and where S is the part of the cone $z^2 = x^2 + y^2$ that lies between the planes $z = 1$ and $z = 2$. The cone can be parametrized by $\vec{r}(u, v) = [v\cos(u), v\sin(u), v]$ with $0 \leq u \leq 2\pi$ and $1 \leq v \leq 2$.



Solution:
The boundary consists of two circles. At $z = 1$ we have the curve

$$\vec{r}(t) = [\cos(t), \sin(t), 1]$$

At $z = 2$ we have

$$\vec{r}(t) = [2\cos(t), -2\sin(t), 2] \ .$$

The second curve is oriented clockwise so that the surface is to the left. By Stokes theorem, we can compute the flux $\int\int_S \text{curl}(\mathbf{F}) \cdot d\vec{S}$ is the sum of the two line integrals

$$\int_0^{2\pi} [-7\sin(t), \cos(t), 1] \cdot [-\sin(t), \cos(t), 0] = dt = 8\pi$$

and

$$\int_0^{2\pi} [14\sin(t), -2\cos(t), -2] \cdot [-2\sin(t), 2\cos(t), 0] \ dt = -32\pi$$

The answer is $\boxed{-24\pi}$.

2 Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x, y, z) = [x^3 - x^5, 7e^x, z^5 + z^9]$ and where C is the boundary of the part of the plane $6x+3y+z = 12$ in the first octant, oriented counterclockwise as viewed from above.

Solution:

Use Stokes theorem. We parametrize the surface as

$$\vec{r}(u,v) = [u, v, 12 - 2u - 3v] \,.$$

We have $\vec{r_u} \times \vec{r_v} = [6, 3, 1]$. The curl of the vector field is $[0, 0, 7e^u]$. Integrating gives

$$\int_0^2 \int_0^{(4-2u} 7e^u \; dvdu$$

The answer is $-42 + 14e^2$.

3 Evaluate the line integral $\int_C \vec{F} \cdot d\mathbf{r}$, where $\vec{F}(x, y, z) = [xy, 2z, 3y]$ and C is the curve of intersection of the plane $x + z = 5$ and the cylinder $x^2 + y^2 = 9$, oriented counterclockwise as viewed from above.

Solution:
The intersection of the plane and the cylinder is an ellipse lying over the disk D given by $x^2 + y^2 \leq 9$ in the xy-plane. We parametrize by $\vec{r}(x, y) = [x, y, 5 - x]$ for (x, y) lying the ellipse D. The usual computation finds

$$\vec{r}_x \times \vec{r}_y = [1, 0, 1] \text{ and } \mathrm{curl}\vec{F} = [1, 0, -x].$$

(Notice that the upward orientation from our parametrization of S is compatible with the orientation of C.) Now we apply Stokes's Theorem, eventually switching to polar coordinates to compute the integral:

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \iint_D \mathrm{curl}\vec{F} \cdot (\vec{r}_x \times \vec{r}_y)\ dx\, dy \\
&= \iint_D [1, 0, -x] \cdot [1, 0, 1]\, dx\, dy = \iint_D (1 - x)\ dx\, dy \\
&= \int_0^{2\pi} \int_0^3 (1 - r\cos\theta)\, r\, dr\, d\theta = \int_0^{2\pi} \left(\frac{9}{2} - 9\cos\theta\right)\ d\theta \\
&= 9\pi.
\end{aligned}$$

4 Compute both sides of Stokes' Theorem for $\vec{F}(x, y, z) = [-2yz, y, 3x]$ and the surface S which is the part of the paraboloid $z = 5 - x^2 - y^2$ that lies above the plane $z = 1$, oriented upwards.

Solution:
The paraboloid intersects the plane $z = 1$ when $1 = 5 - x^2 - y^2 \Rightarrow x^2 + y^2 = 4$, so the boundary curve C is the circle $x^2 + y^2 = 4$, $z = 1$ oriented in the counterclockwise direction as viewed from above. We can parametrize C by $\vec{r}(t) = [2\cos t, 2\sin t, 1]$, $0 \le t \le 2\pi$, and then $\vec{r}'(t) = [-2\sin t, 2\cos t, 0]$. Thus $\vec{F}(\vec{r}(t)) = [-4\sin t, 2\sin t, 6\cos t]$, $\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = [8\sin^2 t, 4\sin t\cos t, 0]$ and

$$\begin{aligned}\oint_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} (8\sin^2 t + 4\cos t \sin t)dt \\ &= 8(\frac{1}{2}t - \frac{1}{4}\sin 2t) + 2\sin^2 t|_0^{2\pi} \\ &= 8\pi\end{aligned}$$

Now curl $\vec{F} = [0, -3 - 2y, 2z]$. The surface $r(x, y) = [x, y, 5 - x^2 - y^2]$ satisfies $r_x \times r_y = [1, 0, -2x] \times [0, 1, -2y] = [2x, 2y, 1]$ for $z5 - x^2 - y^2$, we have

$$\begin{aligned}\iint_S \text{curl}\vec{F} \cdot d\vec{S} &= \iint_D [-0 - (-3 - 2y)(-2y) + 2z]\, dA \\ &= \iint_D [-6y - 4y^2 + 2(5 - x^2 - y^2]\, dA \\ &= \int_0^{2\pi} \int_0^2 [-6r\sin\theta - 4r^2\sin^2\theta + 2(5 - r^2)] r dr d\theta \\ &= \left[-2r^3\sin\theta - r^4\sin^2\theta + 5r^2 - \frac{1}{2}r^4\right]_{r=0}^{r=2} d\theta \\ &= \int_0^{2\pi} (-16\sin\theta - 16\sin^2\theta + 20 - 8)d\theta \\ &= 16\cos\theta - 16\left(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right) + 12\theta|_0^{2\pi} \\ &= 8\pi\end{aligned}$$

5 a) Evaluate $\int_C \vec{F}(x, y, z) \cdot d\vec{r}$ with $\vec{F}(x, y, z) = [y + \sin x, z^2 + \cos y, x^3]$, where C is the curve $\vec{r}(t) = [\sin t, \cos t, \sin 2t], 0 \le t \le 2\pi$ which as you can see lies on the surface $z = 2xy$.

b) Explain without doing any computation that if S is the torus $\vec{r}(u,v) = [(2+\cos(v))\cos(u), (2+\cos(v))\sin(u), \sin(v)]$ with $0 \leq u \leq 2\pi, 0 \leq v \leq 2\pi$ and $\vec{F}$ is a vector field like $\vec{F}(x,y,z) = [e^{e^{e^x}}, \sin\sin(y+z+x), x^{100}]$ then $\int\int_S \text{curl}(\vec{F}) \cdot d\vec{S} = 0$.

Solution:

(a) The curl of $\vec{F}$ is $-4xy, -3x^2, -1]$. The given curve is the boundary of the surface $z = 2xy$ above the unit disk. $D = \{x^2 + y^2 \leq 1\}$. C is traversed clockwise, so that we will have to take care of the orientation. Using Stokes theorem with the surface $\vec{r}(x,y) = [x, y, 2xy]$ which has $r_x \times r_y = [-2y, -2x, 1]$ we have to integrate $F(r(x,y)) \cdot r_x \times r_y = [-4xy, -3x^2, -1] \cdot [-2y, -2x, 1] = 8xy^2 + 6x^3 - 1$ which gives

$$
\begin{aligned}
&= \iint_D (8xy^2 + 6x^3 - 1)\,dA \\
&= \int_0^{2\pi} \int_0^1 (8r^3 \cos(\theta) \sin^2(\theta) + 6r^3 \cos^3(\theta) - 1) r\,dr\,d\theta \\
&= \int_0^{2\pi} (\frac{8}{5} \cos\theta \sin^2\theta + \frac{6}{5} \cos^3\theta - \frac{1}{2})\,d\theta \\
&= -\pi
\end{aligned}
$$

Change the sign because the orientation is off to get π.

(b) The surface is closed, meaning that there is no boundary. When having no boundary, the side of Stokes theorem which contains the line integral is zero. If you like to use Stokes with a aboundary, cut the surface into two (cut the bagle on one circle). You get a surface which is cylinder like and has two boundaries (both circles). But they are oriented differently. The line integral along one is minus the line integral along the other. They cancel. An other possibility is to cut a little hole in the bagle to get a surface with boundary. The line integral along the boundary of the little hole goes to zero if the hole size goes to zero. We will in the last lecture see an other reason why the integral is zero. The divergence theorem will relate the flux with a triple integral.

Main points

Stokes theorem: Let S be a surface bounded by a curve C and $\vec{F}$ be a vector field. Then

$$\int\int_S \text{curl}(\vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} \ .$$

The orientation of S is given by the parametrization: the orientation of C is such that if you walk along C with the head in the "up" direction $\vec{r}_u \times \vec{r}_v$ and your nose into the $\vec{r}\,'$ direction, then your left foot is on the surface.

Written out in detail, we have

$$\int\int_R \text{curl}(\vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) \ dudv = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \ dt$$

A lot of things come together here: surfaces, curves, dot product, cross product, triple scalar product, vector fields, double integrals and curl. What does it mean? From a SMBC cartoon: **"Stokes theorem? Yeah, that's how if you draw a loop around something, you can tell how much swirly is in it."**