Homework 29: Curl, Div and flux

This homework is due Friday, 11/22/2019.

1 Let f be a scalar field and $\vec{F}$ a vector field in space. Determine which expression is meaningful. If not, explain why. If so, state whether it is a scalar field or a vector field. In all problems, we deal with functions and vector fields in 3D space.

a)	$\text{grad}(\vec{F})$	b)	$\text{grad}(\text{div}(\vec{F}))$
c)	$\text{curl}(\text{div}(\vec{F}))$	d)	$\text{curl}(\text{grad}(f))$
e)	$\text{div}(\text{grad}(f))$	f)	$\text{grad}(\text{div}(f))$
g)	$\text{curl}(\text{curl}(\vec{F}))$	h)	$\text{div}(\text{div}(\vec{F}))$
i)	$\text{curl}(f)$	j)	$\text{curl}(\text{div}(f))$
k)	$\text{grad}(f) \times \text{div}(\vec{F})$	l)	$\text{div}(\text{curl}(\text{grad} f))$

2 a) Is there a vector field $\vec{G}(x, y, z)$ such that $\text{curl}(\vec{G}) = [8, 7, 12]$. If yes, find one.

b) Is there a vector field $\vec{G}(x, y, z)$ such that

$$\text{curl}(\vec{G}) = [xyz, -y^2z, x + yz^2] \text{ ?}$$

If yes, find one.

c) Assume $\vec{F}$ is a gradient field. Does this imply that there is a vector field $\vec{G}$ such that $\text{curl}(\vec{G}) = \vec{F}$? If yes, show it. If no, find a counter example.

3 a) Verify that any vector field of the form

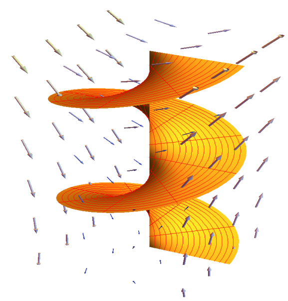
$$\vec{F}(x, y, z) = [f(x), g(y), h(z)]$$

is irrotational (has zero curl everywhere) b) Verify that any vector field of the form

$$\vec{F}(x, y, z) = [f(y, z), g(x, z), h(x, y)]$$

is incompressible (has zero divergence everywhere).
c) Find a non-constant vector field $\vec{F}$ such that $\text{curl}(\vec{F}) = [0, 0, 0]$.
d) Find a non-constant vector field $\vec{F}$ such that $\text{div}(\vec{F}) = 0$.

4 Evaluate the flux integral $\int\int_S \vec{F} \cdot d\vec{S}$ if $\vec{F}(x, y, z) = [7x, y^2, z]$ and S is the helicoid $\vec{r}(u, v) = [u \cos v, u \sin v, v]$, $0 \leq u \leq 1, 0 \leq v \leq 6\pi$ which has an upward orientation.



5 Evaluate the flux integral $\int\int_S \vec{F} \cdot d\vec{S}$ for the vector field

$$\vec{F}(x, y, z) = [x, y, 5] \ ,$$

where S is part of the cylinder $x^2 + z^2 = 1$ that is bound by $y = 0$ and $y = 7$. The surface is oriented outwards.

Main points

The **curl** of a vector field $\vec{F} = [P, Q, R]$ is the vector field $\text{curl}(P, Q, R) = [R_y - Q_z, P_z - R_x, Q_x - P_y]$. In 2 dimensions, the curl of $\vec{F} = [P, Q]$ is the scalar $Q_x - P_y$. The **divergence** of $\vec{F}(x, y, z) = [P, Q, R]$ is $\text{div}(\vec{F})(x, y, z) = P_x(x, y, z) + Q_y(x, y, z) + R_z(x, y, z)$. The **flux integral** of $\vec{F}$ through S parametrized by $\vec{r}(u, v) = [x(u, v), y(u, v), z(u, v)]$ is

$$\int\int_G \vec{F}(\vec{r}(u, v)) \cdot (\vec{r}_u \times \vec{r}_v) \ dudv \ .$$