Homework 29: Curl, Div and flux

This homework is due Friday, 11/22/2019.

1 Let f be a scalar field and $\vec{F}$ a vector field in space. Determine which expression is meaningful. If not, explain why. If so, state whether it is a scalar field or a vector field. In all problems, we deal with functions and vector fields in 3D space.

a) $\text{grad}(\vec{F})$	b) $\text{grad}(\text{div}(\vec{F}))$
c) $\text{curl}(\text{div}(\vec{F}))$	d) $\text{curl}(\text{grad}(f))$
e) $\text{div}(\text{grad}(f))$	f) $\text{grad}(\text{div}(f))$
g) $\text{curl}(\text{curl}(\vec{F}))$	h) $\text{div}(\text{div}(\vec{F}))$
i) $\text{curl}(f)$	j) $\text{curl}(\text{div}(f))$
k) $\text{grad}(f) \times \text{div}(\vec{F})$	l) $\text{div}(\text{curl}(\text{grad} f))$

Solution:

(a) $\text{grad}(\vec{F})$ is meaningless because $\vec{F}$ is not a scalar field.

(b) $\text{grad}(\text{div}(\vec{F})$ is a vector field.

(c) $\text{div}\vec{F}$ is a scalar field, the curl is not defined.

(d) $\text{curl}(\text{grad}(f))$ is a vector field.

(e) $\text{div}(\text{grad}(f))$ is a scalar field.

(f) $\text{grad}(\text{div}(f))$ is meaningless because f is a scalar field.

(g) $\text{curl}(\text{curl}(\vec{F}))$ is a vector field.

(h) $\text{div}(\text{div}(\vec{F}))$ is meaningless because $\text{div}\vec{F}$ is a scalar field.

(i) $\text{div}(\text{curl}(\text{grad}f))$ is a scalar field.

(j) $\text{curl}f = \nabla \times f$ is meaningless because f is a scalar field.

(k) $\text{grad}(f) \times \text{div}(\vec{F})$ is meaningless because $\text{div}\vec{F}$ is a scalar field.

(l) The divergence of f is not defined.

2 a) Is there a vector field $\vec{G}(x, y, z)$ such that $\text{curl}(\vec{G}) = [8, 7, 12]$. If yes, find one.
b) Is there a vector field $\vec{G}(x, y, z)$ such that

$$\text{curl}(\vec{G}) = [xyz, -y^2z, x + yz^2] \ ?$$

If yes, find one.
c) Assume $\vec{F}$ is a gradient field. Does this imply that there is a vector field $\vec{G}$ such that $\text{curl}(\vec{G}) = \vec{F}$? If yes, show it. If no, find a counter example.

Solution:

a) Yes, for example $[-12y, -8z, -7x]$. b) No. Assume there is such a G. Then $\text{div}(\text{curl}\vec{G}) = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(-y^2 z) + \frac{\partial}{\partial z}(yz^2) = yz - 2yz + 2yz = yz \neq 0$ which contradicts Theorem 11.

c) We necessarily need the divergence to be zero. Take $[x, 0, 0]$ for example.

3 a) Verify that any vector field of the form

$$\vec{F}(x, y, z) = [f(x), g(y), h(z)]$$

is irrotational (has zero curl everywhere) b) Verify that any vector field of the form

$$\vec{F}(x, y, z) = [f(y, z), g(x, z), h(x, y)]$$

is incompressible (has zero divergence everywhere).

c) Find a non-constant vector field $\vec{F}$ such that $\text{curl}(\vec{F}) = [0, 0, 0]$.

d) Find a non-constant vector field $\vec{F}$ such that $\text{div}(\vec{F}) = 0$.

Solution:

(a) $\text{curl}\vec{F} = \left(\frac{\partial h(z)}{\partial y} - \frac{\partial g(y)}{\partial z}\right)\vec{i} + \left(\frac{\partial f(x)}{\partial z} - \frac{\partial h(z)}{\partial x}\right)\vec{j} + \left(\frac{\partial g(y)}{\partial x} - \frac{\partial f(x)}{\partial y}\right)\vec{k} = (0-0)\vec{i} + (0-0)\vec{j} + (0-0)\vec{k} = 0.$

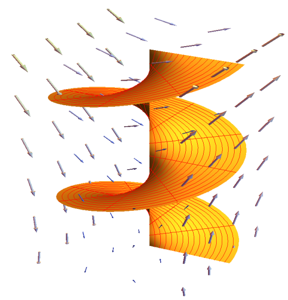
Hence $\vec{F}$ is irrotational.

(b) $\text{div}\vec{F} = \frac{\partial(f(y,z))}{\partial x} + \frac{\partial(g(x,z))}{\partial y} + \frac{\partial(h(x,y))}{\partial z} = 0$ so $\vec{F}$ is incompressible.

(c) Take a gradient field.

(d) Take a field $\text{curl}(G)$.

4 Evaluate the flux integral $\int\int_S \vec{F} \cdot d\vec{S}$ if $\vec{F}(x,y,z) = [7x, y^2, z]$ and S is the helicoid $\vec{r}(u,v) = [u\cos v, u\sin v, v]$, $0 \le u \le 1, 0 \le v \le 6\pi$ which has an upward orientation.



Solution:
Compute $F(r(u,v))$ and $r_u \times r_v$ then take the dot product. We get then the integral $\int_0^1 \int_0^{6\pi} -u^2\sin^2(v)\cos(v) + v\left(u\sin^2(v) + u\cos^2(v)\right) + 7u\sin(v)\cos(v)\ dudv$. This simplifies to $\int_0^1 \int_0^{6\pi} uv\ dudv$ as the odd trig functions functions in v become zero. the result is $9\pi^2$.

5 Evaluate the flux integral $\int\int_S \vec{F} \cdot d\vec{S}$ for the vector field

$$\vec{F}(x,y,z) = [x, y, 5] \ ,$$

where S is part of the cylinder $x^2 + z^2 = 1$ that is bound by $y = 0$ and $y = 7$. The surface is oriented outwards.

Solution:
Parametrize the surface S: $\vec{F}(\vec{r}(\theta, y)) = [\sin\theta, y, 5]$ and $\vec{r}_\theta \times \vec{r}_y = [\sin\theta, 0, \cos\theta] \Rightarrow$

$$\iint_S \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^7 \left(\sin^2\theta + 5\cos\theta\right)\, dy\, d\theta = 7\pi \ .$$

Main points

The **curl** of a vector field $\vec{F} = [P, Q, R]$ is the vector field $\text{curl}(P, Q, R) = [R_y - Q_z, P_z - R_x, Q_x - P_y]$. In 2 dimensions, the curl of $\vec{F} = [P, Q]$ is the scalar $Q_x - P_y$. The **divergence** of $\vec{F}(x, y, z) = [P, Q, R]$ is $\text{div}(\vec{F})(x, y, z) = P_x(x, y, z) + Q_y(x, y, z) + R_z(x, y, z)$. The **flux integral** of $\vec{F}$ through S parametrized by $\vec{r}(u, v) = [x(u, v), y(u, v), z(u, v)]$ is

$$\int\int_G \vec{F}(\vec{r}(u, v)) \cdot (\vec{r}_u \times \vec{r}_v) \, dudv \ .$$