Homework 28: Green's theorem

This homework is due Wednesday, 11/20/2019.

1 Evaluate the line integral

$$\int_C \vec{F} \cdot d\vec{r} \ ,$$

where $\vec{F}(x,y) = [30y + 3x^{15} - 1, 5x + \cos(9y)y^{300}]$ and C consists of the line segments from $(0,2)$ to $(0,0)$ and from (0,0) to (2,0) and the curve $y = \sqrt{4-x^2}$ from $(2,0)$ to $(0,2)$.

2 a) Find the line integral $\int_C \vec{F} \cdot d\vec{r}$ with

$\vec{F}(x,y) = [y^2\cos(x) + \sin(\sin(x)), 45x + x^3 + 2y\sin(x) + \sin(\sin(y))]$

where C is the triangular path from (0,0) to (2,6) to (2,0) to (0,0). Watch the orientation of the curve!
b) Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where

$$\vec{F}(x,y) = [-8y + 1/(1+x+y^3), 7x + 3y^2/(1+x+y^3)]$$

and C is the circle $(x-6)^2 + (y-7)^2 = 49$ oriented counterclockwise.

3 Compute the area of the region bounded by the **distorted hypocycloid**

$$\vec{r}(t) = [5\cos^3(t), 16\sin^3(t)], 0 \leq t \leq 2\pi \ .$$

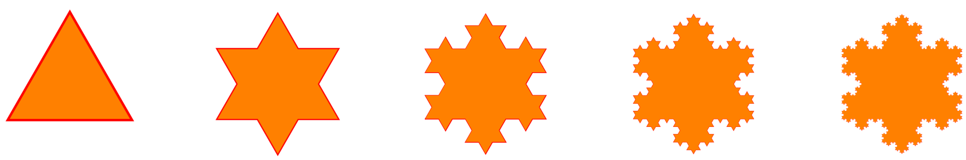
We can not do that directly but we can use a theorem!

4 Calculate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x,y) = [x^2 + y, 3x - \sin(y^2)]$ and C is the ellipse $x^2/100 + y^2/16 = 1$ oriented clockwise.

5 Use Green's Theorem to evaluate

$$\int_C [\sin(\sqrt{1+x^3}) + 21y, 121x] \, d\vec{r} \ ,$$

where C is the boundary of the region $K(4)$. You see in the picture $K(0), K(1), K(2), K(3), K(4)$. The first $K(0)$ is an equilateral triangle of length 1. The second $K(1)$ is $K(0)$ with 3 equilateral triangles of length 1/3 added. $K(2)$ is $K(1)$ with $3 * 4^1$ equilateral triangles of length 1/9 added. $K(3)$ is $K(2)$ with $3 * 4^2$ of length 1/27 added and $K(4)$ is $K(3)$ with $3*4^3$ triangles of length 1/81 added. Remark. We could now find the line integral in the limit $K = K(\infty)$, a **fractal** called the **Koch snowflake** It has dimension $\log(4)/\log(3) = 1.26\ldots$ which is between 1 and 2.



Main points

The **curl** of a vector field $\vec{F}(x,y) = [P(x,y), Q(x,y)]$ is the scalar field

$$\text{curl}(F)(x,y) = Q_x(x,y) - P_y(x,y) \; .$$

Green's theorem: If $\vec{F}(x,y) = [P(x,y), Q(x,y)]$ is a vector field and G is a region for which the boundary C is parametrized so that R is "to the left", then

$$\int_C \vec{F} \cdot \vec{dr} = \iint_G \text{curl}(F) \; dxdy \; .$$

Extra credit: Area Computation using data

Green's theorem shows that for $\vec{F}(x,y) = [-y,x]/2$, the area of a region G with boundary curve C is $\int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt$. The area of a polygon by parametrizing its line segments.

a) Compute $\int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt$ if C is the line segment starting at $P = (a,b)$ and ending at $Q = (c,d)$.

b) Use a) and Green's theorem to get the area of the quadrilateral G with boundary points $(1,0)$, $(4,3)$, $(0,2)$, $(1,1)$.

c) Verify that $A = \Sigma_{k=1}^{n}(a_k d_k - b_k c_k)/2$ is the area of the polygon with n segments (a_k, b_k) to (c_k, d_k).

d) Mathematica has stored areas of country data and polygonal approximations of the countries. Use the code below (you can copy paste from the "data" part of the website!) to pick a country and report its area. You might see a negative value when using the above formula. What could be the reason?

e) Compare your area with the area given by Wikipedia.


```
country=RandomChoice[CountryData[]]
CountryData[country, "LandArea"]
A = First[CountryData[country,"SchematicCoordinates"]];
P[{x_,y_}] := 6371*{y*Pi/180,Sin[x*Pi/180]};
B = Map[P,A];
Area[Polygon[B]]
MyArea[A_]:=Sum[n=Length[A]; a=A[[k,1]]; b=A[[k,2]];
  c=A[[Mod[k,n]+1,1]]; d=A[[Mod[k,n]+1,2]]; (a*d - b*c)/2,{k,n}];
MyArea[B]
```