Homework 28: Green's theorem

This homework is due Wednesday, 11/20/2019.

1 Evaluate the line integral

$$\int_C \vec{F} \cdot d\vec{r} \ ,$$

where $\vec{F}(x,y) = [30y + 3x^{15} - 1, 5x + \cos(9y)y^{300}]$ and C consists of the line segments from $(0,2)$ to $(0,0)$ and from (0,0) to (2,0) and the curve $y = \sqrt{4-x^2}$ from $(2,0)$ to $(0,2)$.

Solution:
We have $Q_x - P_y = 5 - 30 = -15$ so that the result is -25 times the area of a quarter circle with radius 2 which has area π. The result is -25π.

2 a) Find the line integral $\int_C \vec{F} \cdot d\vec{r}$ with

$$\vec{F}(x,y) = [y^2\cos(x) + \sin(\sin(x)), 45x + x^3 + 2y\sin(x) + \sin(\sin(y))]$$

where C is the triangular path from (0,0) to (2,6) to (2,0) to (0,0). Watch the orientation of the curve!
b) Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where

$$\vec{F}(x,y) = [-8y + 1/(1+x+y^3), 7x + 3y^2/(1+x+y^3)]$$

and C is the circle $(x-6)^2 + (y-7)^2 = 49$ oriented counterclockwise.

Solution:
a) The curl is $45 + 3x^2$. We have to integrate this function over the triangle.

$$\int_0^2 \int_0^{3x} (45 + 3x^2)\ dydx\ .$$

The result of this integration is 306. Since the orientation is opposite, the result is -306.
b) The curl is 15. The result is the area of the disk times 15 which is $15 \cdot 49\pi = 735\pi$.

3 Compute the area of the region bounded by the **distorted hypocycloid**

$$\vec{r}(t) = [5\cos^3(t), 16\sin^3(t)], 0 \le t \le 2\pi\ .$$

We can not do that directly but we can use a theorem!

Solution:
Take a vector field $\vec{F}(x, y) = [0, x]$ which has the curl 1. Then by Green the area is the line integral

$$\begin{aligned} & \int_0^{2\pi} [0, 5\cos^3(t)] \cdot [-15\cos^2(t)\ \sin(t), 48\sin^2(t)\cos(t)]\ dt \\ = \ & 240 \int_0^{2\pi} \cos^4(t)\sin^2(t)\ dt \\ = \ & 240 \int_0^{2\pi} \left(\cos^2(t)\sin^2(t)\right)(\cos^2(t))\ dt \\ \Rightarrow \ & 30\pi\ . \end{aligned}$$

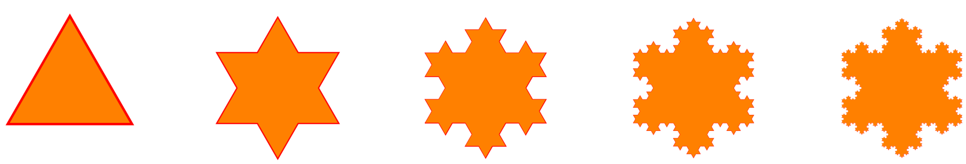
4 Calculate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x, y) = [x^2 + y, 3x - \sin(y^2)]$ and C is the ellipse $x^2/100 + y^2/16 = 1$ oriented clockwise.

Solution:
By Green's Theorem we have 2 times the area 40π of the ellipse. But because the curve is oriented clockwise, the result is -80π.

5 Use Green's Theorem to evaluate

$$\int_C [\sin(\sqrt{1+x^3}) + 21y, 121x] \, d\vec{r} \, ,$$

where C is the boundary of the region $K(4)$. You see in the picture $K(0), K(1), K(2), K(3), K(4)$. The first $K(0)$ is an equilateral triangle of length 1. The second $K(1)$ is $K(0)$ with 3 equilateral triangles of length $1/3$ added. $K(2)$ is $K(1)$ with $3 * 4^1$ equilateral triangles of length $1/9$ added. $K(3)$ is $K(2)$ with $3 * 4^2$ of length $1/27$ added and $K(4)$ is $K(3)$ with $3*4^3$ triangles of length $1/81$ added. Remark. We could now find the line integral in the limit $K = K(\infty)$, a **fractal** called the **Koch snowflake** It has dimension $\log(4)/\log(3) = 1.26\ldots$ which is between 1 and 2.



Solution:
Since $\text{curl}(F) = 100$, we have to compute the area of $K(3)$ and multiply by 100. We have $|K(3)| = \left(\frac{\sqrt{3}}{4}\right)\left(1+\frac{3}{9}+\frac{12}{81}+\frac{48}{729}+\frac{192}{(9*729)}\right) = 4.778$.
This was not required: in the limit we have a geometric series $|K| = \frac{\sqrt{3}}{4}\left(1+\frac{3}{9}\sum_{k=0}^{\infty}\frac{4^k}{9^k}\right) = \frac{\sqrt{3}}{4}\left(1+\frac{3}{9}\left(\frac{1}{1-\frac{4}{9}}\right)\right) = \frac{2\sqrt{3}}{5}$.

Main points

The **curl** of a vector field $\vec{F}(x,y) = [P(x,y), Q(x,y)]$ is the scalar field

$$\text{curl}(F)(x,y) = Q_x(x,y) - P_y(x,y) \ .$$

Green's theorem: If $\vec{F}(x,y) = [P(x,y), Q(x,y)]$ is a vector field and G is a region for which the boundary C is parametrized so that R is "to the left", then

$$\int_C \vec{F} \cdot \vec{dr} = \iint_G \text{curl}(F) \ dxdy \ .$$

Extra credit: Area Computation using data

Green's theorem shows that for $\vec{F}(x,y) = [-y, x]/2$, the area of a region G with boundary curve C is $\int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$. The area of a polygon by parametrizing its line segments.

a) Compute $\int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$ if C is the line segment starting at $P = (a, b)$ and ending at $Q = (c, d)$.

b) Use a) and Green's theorem to get the area of the quadrilateral G with boundary points $(1, 0)$, $(4, 3)$, $(0, 2)$, $(1, 1)$.

c) Verify that $A = \Sigma_{k=1}^{n}(a_k d_k - b_k c_k)/2$ is the area of the polygon with n segments (a_k, b_k) to (c_k, d_k).

d) Mathematica has stored areas of country data and polygonal approximations of the countries. Use the code below (you can copy paste from the "data" part of the website!) to pick a country and report its area. You might see a negative value when using the above formula. What could be the reason?

e) Compare your area with the area given by Wikipedia.


```
country=RandomChoice[CountryData[]]
CountryData[country, "LandArea"]
A = First[CountryData[country,"SchematicCoordinates"]];
P[{x_, y_}] := 6371*{y*Pi/180,Sin[x*Pi/180]};
B = Map[P,A];
Area[Polygon[B]]
MyArea[A_]:=Sum[n=Length[A]; a=A[[k,1]]; b=A[[k,2]];
   c=A[[Mod[k,n]+1,1]]; d=A[[Mod[k,n]+1,2]]; (a*d - b*c)/2,{k,n}];
MyArea[B]
```