Homework 27: Theorem of line integrals

This homework is due Monday, 11/18.

1 a) Find a function f such that $\vec{F} = \nabla f$ if

$$\vec{F}(x,y) = [9x^8 - 2xy^2 + 3y, 6y^5 - 2x^2y + 3x] \ .$$

b) Use a) to evaluate $\int_C \vec{F} \cdot d\vec{r}$ along the parabola $y = 2x^2$ from $(-1,2)$ to $(2,8)$.
c) What is $\int_C \vec{F} \cdot d\vec{r}$ if $\vec{r}(t) = [\cos(8t), \sin(12t)]$ with $t \in [0,\pi]$?

2 a) Find a function f such that $\vec{F} = \nabla f$ if $\vec{F}(x,y) = [70y^3/(1+x^2), 210y^2 \arctan(x)]$.
b) Use a) to evaluate $\int_C \vec{F} \cdot d\vec{r}$ along the curve $\vec{r}(t) = [t^8, 2t^6]$ with $0 \leq t \leq 1$.

3 We look at the vector field

$$\vec{F}(x,y,z) = [P,Q,R] = [\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 4z]$$

which is defined on $G = \{\mathbf{R}^3 \mid x^2+y^2 \neq 0\}$ (which is space without the z-axes).
a) Check that $[R_y - Q_z, P_z - R_x, Q_x - P_y] = [0,0,0]$ everywhere on G.
b) Compute the line integral of $\vec{F}$ along the closed loop $\vec{r}(t) = [\cos(t), \sin(t), 0]$ with $0 \leq t \leq 2\pi$.

4 A force field $\vec{F}$ consists of gravitational force combined with wind forces: $\vec{F}(x,y,z) = [\sin(x), \cos(y), -10+z]$. The path is given by $\vec{r}(t) = [5t, t, 30 - \sin(t)/10]$, where $0 \leq t \leq \pi$. Compute the work $\int_0^\pi \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt$ done in this field.

5 a) Confirm that the vector field $\vec{F}(x, y, z) = [y, x, xyz]$ is not conservative.

b) Find two different curves C_1, C_2 from $(0, 0, 0)$ to $(1, 1, 0)$ for which the line integrals of $\vec{F}$ along C_1, C_2 are different.

Main points

This theorem is the first generalization of the fundamental theorem of calculus to higher dimensions. It tells that the work done along a path is the potential energy difference.

Fundamental theorem of line integrals: If $\vec{F} = \nabla f$, then

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt = f(\vec{r}(b)) - f(\vec{r}(a)) \; .$$

This theorem can be used to dramatically simplify the computation of a line integral. Just find the potential f and evaluate the difference of potential values. Recall that a region R is called **simply connected** if every closed loop in R can be pulled together to a point within R.

The three concepts "gradient field", "closed loop property" and "conservative" are the same:

Gradient field $\leftrightarrow$ Conservative $\leftrightarrow$ Closed loop property

In simply connected open regions, these three properties are all equivalent to being irrotational $\mathrm{curl}(\vec{\mathrm{F}}) = \mathrm{Q_x} - \mathrm{P_y} = 0$. For gradient fields $\vec{F} = [P, Q, R] = [f_x, f_y, f_z] = \nabla f$ we have zero curl $\mathrm{curl}(\vec{F}) = [R_y - Q_z, P_z - R_x, Q_x - P_y] = [0, 0, 0]$. We will see next week that zero curl everywhere implies the closed loop property in a simply connected region.