Homework 27: Theorem of line integrals

This homework is due Monday, 11/18.

1 a) Find a function f such that $\vec{F} = \nabla f$ if

$$\vec{F}(x,y) = [9x^8 - 2xy^2 + 3y, 6y^5 - 2x^2y + 3x] .$$

b) Use a) to evaluate $\int_C \vec{F} \cdot d\vec{r}$ along the parabola $y = 2x^2$ from $(-1,2)$ to $(2,8)$.
c) What is $\int_C \vec{F} \cdot d\vec{r}$ if $\vec{r}(t) = [\cos(8t), \sin(12t)]$ with $t \in [0,\pi]$?

Solution:
(a) $f(x,y) = x^9 - x^2 * y^2 + y^6 + 3xy$.
(b) The integral equals $f(2,8) - f(-1,2) = 262395$.
(c) The integral is zero. It is a closed loop.

2 a) Find a function f such that $\vec{F} = \nabla f$ if $\vec{F}(x,y) = [70y^3/(1+x^2), 210y^2 \arctan(x)]$.
b) Use a) to evaluate $\int_C \vec{F} \cdot d\vec{r}$ along the curve $\vec{r}(t) = [t^8, 2t^6]$ with $0 \le t \le 1$.

Solution:
(a) $f(x,y) = 70y^3 \arctan x$,
(b) The integral is $f(1,2) - f(0,0) = 140\pi - 0 = 140\pi$.

3 We look at the vector field

$$\vec{F}(x,y,z) = [P,Q,R] = [\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 4z]$$

which is defined on $G = \{\mathbf{R}^3 \mid x^2 + y^2 \neq 0\}$ (which is space without the z-axes).
a) Check that $[R_y - Q_z, P_z - R_x, Q_x - P_y] = [0, 0, 0]$ everywhere on G.
b) Compute the line integral of $\vec{F}$ along the closed loop $\vec{r}(t) = [\cos(t), \sin(t), 0]$ with $0 \leq t \leq 2\pi$.

Solution:
a) This is obtained directly by differentiation.
b) The result is 2π. We have zero curl everywhere, but no closed loop property.
This region R is not simply connected.

4 A force field $\vec{F}$ consists of gravitational force combined with wind forces: $\vec{F}(x, y, z) = [\sin(x), \cos(y), -10 + z]$. The path is given by $\vec{r}(t) = [5t, t, 30 - \sin(t)/10]$, where $0 \leq t \leq \pi$. Compute the work $\int_0^\pi \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$ done in this field.

Solution:
Best use the FTLI. The potential is $f(x, y, z) = -\cos(x) + \sin(y) - 10z + z^2/2$. Instead of integrating, we just have to evaluate $f(\vec{r}(\pi)) - f(\vec{r}(0)) = f(5\pi, \pi, 30) - f(0, 0, 30) = 2$. It was also possible to do the line integral directly, but it was considerably more work.

5 a) Confirm that the vector field $\vec{F}(x, y, z) = [y, x, xyz]$ is not conservative.
b) Find two different curves C_1, C_2 from $(0, 0, 0)$ to $(1, 1, 0)$ for which the line integrals of $\vec{F}$ along C_1, C_2 are different.

Solution:
(a) For example, $P_z = 0 \neq yz = R_x$.
(b) Unfortunately, we cannot take both curves that lie entirely in the xy-plane since $P_y = 1 = Q_x$. Consider a vertical line segment $I_{x,y}$ which joins $(x, y, 0)$ to $(x, y, 1)$. Then the integral over this segment is $\int_0^1 xyz dz = xy/2$. This gives us a hint how to proceed. Take γ_1 to be any curve from $(0, 0, 0)$ to $(1, 1, 0)$ in the xy-plane. Set γ_2 be the union of $I_{0,0}$ (oriented upwards), a curve in the plane $z = 1$ above γ_1 and $I_{1,1}$ (oriented downwards). The difference between the integrals over γ_2 and γ_1 is $xy/2|_{1,1} - xy/2|_{0,0} = 1/2$ so the two integrals are not the same.

Main points

This theorem is the first generalization of the fundamental theorem of calculus to higher dimensions. It tells that the work done along a path is the potential energy difference.

Fundamental theorem of line integrals: If $\vec{F} = \nabla f$, then

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = f(\vec{r}(b)) - f(\vec{r}(a)) \, .$$

This theorem can be used to dramatically simplify the computation of a line integral. Just find the potential f and evaluate the difference of potential values. Recall that a region R is called **simply connected** if every closed loop in R can be pulled together to a point within R.

The three concepts "gradient field", "closed loop property" and "conservative" are the same:

Gradient field $\leftrightarrow$ Conservative $\leftrightarrow$ Closed loop property

In simply connected open regions, these three properties are all equivalent to being irrotational $\mathrm{curl}(\vec{\mathrm{F}}) = \mathrm{Q_x} - \mathrm{P_y} = 0$. For gradient fields $\vec{F} = [P, Q, R] = [f_x, f_y, f_z] = \nabla f$ we have zero curl $\mathrm{curl}(\vec{F}) = [R_y - Q_z, P_z - R_x, Q_x - P_y] = [0, 0, 0]$. We will see next week that zero curl everywhere implies the closed loop property in a simply connected region.