Homework 26: Line integrals

This homework is due Friday, 11/15

1 a) Evaluate $\int_C \vec{F} \cdot d\vec{r}$ if $\vec{F}(x,y,z) = 7[x^3 + y^4, y - z, z^2 + 1]$ and $\vec{r}(t) = [t^2, t^3, t]$ with $0 \leq t \leq 5$.
b) Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ if $\vec{F}(x,y,z) = [5z, y, -x]$ and $\vec{r}(t) = [2t, \sin(t), \cos(t)],\ 0 \leq t \leq 3\pi$.

2 Determine from each of the following cases, whether $\vec{F}$ is conservative (a gradient field) or not. If it conservative, find a potential function f such that $\vec{F} = \nabla f$.

a) $\vec{F}(x,y,z) = [6x^5 + y, 6y^5 + x, 2z + 5]$
b) $\vec{F}(x,y) = [y + 4x^3, -x + y^5]$
c) $\vec{F}(x,y) = [x + 7e^x \sin(y), y^4 + 7e^x \cos(y)]$
d) $\vec{F}(x,y,z) = [x + yz, y + xz, z^5 - \sin(z) + yz]$

3 An electric current I produces a magnetic field $\vec{B}$ whose flow lines circle the wire. Let $C : [r\cos(t), r\sin(t), 0]$. Ampères law is $\int_C \vec{B} \cdot d\vec{r} = \mu_0 I$, where μ_0 is a constant called permeability. a) Find an expression for a vector field $\vec{B}$ whose flow lines are horizontal circles centered around the z-axis and whose magnitude at a distance r from the z-axis is $B(r)$. b) Using Ampere's law, show that $B(r) = \frac{\mu_0 I}{2\pi r}$. Note that B is a scalar function while $\vec{B}$ is a vector field.

4 Evaluate $\int_C [1 - ye^{-x}, e^{-x}] \cdot d\vec{r}$, where C is the path $r(\vec{t}) = [t, 1 + t + \sin(\sin(t))]$ with $0 \leq t \leq \pi$. You encounter difficulties evaluating the integral. An oracle tells you that you can compute the integral also in a different way: find a function f which is a potential to the vector field, then evaluate $f(\vec{r}(\pi)) - f(\vec{r}(0))$. Use this without justification for now.

5 The topological notions appearing here play a role when deciding whether Clairaut can be reversed. Determine whether or not the given set is (a) open, (b) connected, and (c) simply-connected.
a) $\{(x,y)\,|\,2 < y < 3\}$, b) $\{(x,y)\,|\,2 < |x| < 5\}$
c) $\{(x,y)\,|\,2 \leq x^2 + y^2 \leq 16, y \geq 0\}$, d) $\{(x,y)\,|\,(x,y) \neq (4,5)\}$
e) $\{(x,y,z)\,|\,(x,y,z) \neq \{(\cos(t), \sin(t), 0)\ |\ 0 \leq t \leq 2\pi\}$
f) $\{(x,y,z)\,|\,(x,y,z) \neq (7,8,1)\}$

Main definitions

If $\vec{F}$ is a vector field and $C : t \mapsto \vec{r}(t)$ is a curve defined on the interval $[a,b]$ then $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt$ is the **line integral** of $\vec{F}$ along the curve C. The field $\vec{F}$ is **conservative** in a region R if the line integral from A to B is path independent. It has the **closed loop property** if the line integral along any closed loop is zero. Conservative, path independence and closed loop property are all equivalent. $\vec{F}$ is **irrotational** if $\mathrm{curl}(F) = Q_x - P_y$ is zero everywhere in R. The **Clairaut test**: Zero curl is necessary for a gradient field. For $\vec{F} = [P,Q]$ it is $Q_x - P_y = 0$, in the case $\vec{F} = [P,Q,R]$ it is $[R_y - Q_z, P_z - R_x, Q_x - P_y] = [0,0,0]$.

A subset G of the plane is **open** if every point (x,y) in G is contained in a small disc D centered at (x,y) and $D \subset G$. It is **connected**, if any two points in G can be connected with a curve within G. It is **simply connected** if it is connected and every closed curve in G can be deformed to a point within G.