# Homework 26: Line integrals

This homework is due Friday, 11/15

1 a) Evaluate $\int_C \vec{F} \cdot d\vec{r}$ if $\vec{F}(x,y,z) = 7[x^3 + y^4, y - z, z^2 + 1]$ and $\vec{r}(t) = [t^2, t^3, t]$ with $0 \le t \le 5$.
b) Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ if $\vec{F}(x,y,z) = [5z, y, -x]$ and $\vec{r}(t) = [2t, \sin(t), \cos(t)],\ 0 \le t \le 3\pi$.

**Solution:**
(a) The integral is

$$\int_0^5 7[t^6 + t^9, -t + t^4, 1 + t^2] \cdot [2t, 3t^2, 1]dt =$$

This evaluates to 18312752855/3 (b) The integral is

$$\int_0^\pi [5\cos(t), \sin(t), -2t] \cdot [2, \cos(t), -\sin(t)]dt =$$

$$= \int_0^{3\pi} (10Cos[t] + 2tSin[t] + Cos[t]Sin[t])dt = 6\pi\ .$$

2 Determine from each of the following cases, whether $\vec{F}$ is conservative (a gradient field) or not. If it conservative, find a potential function $f$ such that $\vec{F} = \nabla f$.

a) $\vec{F}(x,y,z) = [6x^5 + y, 6y^5 + x, 2z + 5]$

b) $\vec{F}(x,y) = [y + 4x^3, -x + y^5]$

c) $\vec{F}(x,y) = [x + 7e^x\sin(y), y^4 + 7e^x\cos(y)]$

d) $\vec{F}(x,y,z) = [x + yz, y + xz, z^5 - \sin(z) + yz]$

**Solution:**
Check the Clairaut property $Q_x - P_y$ and integrate if necessary. In the case of $d)$ we have to check all three conditions $Q_x - P_y, R_y - Q_z, P_z - R_x$ or simply provide an example of a function $f$.
a) yes b) no, c) yes, d) no The potential function in a) is $x^6 + y^6 + xy + z^2 + 5z$.
The potential function in c) is $x^2/2 + e^x \sin(y) + y^5/5$.

3 An electric current $I$ produces a magnetic field $\vec{B}$ whose flow lines circle the wire. Let $C : [r\cos(t), r\sin(t), 0]$. Ampères law is $\int_C \vec{B} \cdot d\vec{r} = \mu_0 I$, where $\mu_0$ is a constant called permeability. a) Find an expression for a vector field $\vec{B}$ whose flow lines are horizontal circles centered around the z-axis and whose magnitude at a distance $r$ from the z-axis is $B(r)$. b) Using Ampere's law, show that $B(r) = \frac{\mu_0 I}{2\pi r}$. Note that $B$ is a scalar function while $\vec{B}$ is a vector field.

**Solution:**
The vector field has the form $\vec{B} = B(r)[-y/r, x/r]$ where $B$ depends only on the radius $r$. Above $[-y/r, x/r]$ is the unit vector tangent to a circle of radius $r$. Integrating over a concentric circle of radius $r$, we get: $\int_{C(r)} \vec{B} \cdot d\vec{r} = 2\pi B(r)$. Solving for $B(r)$, we find that $B(r) = \mu_0 I/(2\pi r)$ as desired.

4 Evaluate $\int_C [1 - ye^{-x}, e^{-x}] \cdot d\vec{r}$, where $C$ is the path $r(\vec{t}) = [t, 1+t+\sin(\sin(t))]$ with $0 \leq t \leq \pi$. You encounter difficulties evaluating the integral. An oracle tells you that you can compute the integral also in a different way: find a function $f$ which is a potential to

the vector field, then evaluate $f(\vec{r}(\pi)) - f(\vec{r}(0))$. Use this without justification for now.

**Solution:**
Check first that the vector field is conservative:

$$\frac{\partial(1 - ye^{-x})}{\partial y} = -e^{-x} = \frac{\partial(e^{-x})}{\partial x}.$$

The function $f = x + ye^{-x}$ satisfies $\nabla f = [1 - ye^{-x}, e^{-x}]$, hence the answer should be $f(\pi, 1+\pi) - f(0,1) = e^{-\pi} + \pi + \pi e^{-\pi} - 1$.

5 The topological notions appearing here play a role when deciding whether Clairaut can be reversed. Determine whether or not the given set is (a) open, (b) connected, and (c) simply-connected.
a) $\{(x, y) \,|\, 2 < y < 3\}$, b) $\{(x, y) \,|\, 2 < |x| < 5\}$
c) $\{(x, y) \,|\, 2 \leq x^2 + y^2 \leq 16, y \geq 0\}$, d) $\{(x, y) \,|\, (x, y) \neq (4, 5)\}$
e) $\{(x, y, z) \,|\, (x, y, z) \neq \{(\cos(t), \sin(t), 0) \mid 0 \leq t \leq 2\pi\}$
f) $\{(x, y, z) \,|\, (x, y, z) \neq (7, 8, 1)\}$

**Solution:**

a) The set is a horizontal strip which is open, connected and simply-connected.

b) The set is open but is not connected because it has two components, one where $x$ is positive, and one where $x$ is negative.

c) The set is a semi-annulus. It is connected and simply-connected; but since we have included the boundary, it is not open.

d) The set is a plane minus a point. It is open and connected; but since it has a hole at $(2, 3)$, it is not simply-connected.

e) The set is connected but not simply connected. f) The set **is** simply connected. You can pull together any string to a point.

## Main definitions

If $\vec{F}$ is a vector field and $C : t \mapsto \vec{r}(t)$ is a curve defined on the interval $[a, b]$ then $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt$ is the **line integral** of $\vec{F}$ along the curve $C$. The field $\vec{F}$ is **conservative** in a region $R$ if the line integral from $A$ to $B$ is path independent. It has the **closed loop property** if the line integral along any closed loop is zero. Conservative, path independence and closed loop property are all equivalent. $\vec{F}$ is **irrotational** if $\text{curl}(F) = Q_x - P_y$ is zero everywhere in $R$. The **Clairaut test**: Zero curl is necessary for a gradient field. For $\vec{F} = [P, Q]$ it is $Q_x - P_y = 0$, in the case $\vec{F} = [P, Q, R]$ it is $[R_y - Q_z, P_z - R_x, Q_x - P_y] = [0, 0, 0]$.

A subset $G$ of the plane is **open** if every point $(x, y)$ in $G$ is contained in a small disc $D$ centered at $(x, y)$ and $D \subset G$. It is **connected**, if any two points in $G$ can be connected with a curve within $G$. It is **simply connected** if it is connected and every closed curve in $G$ can be deformed to a point within $G$.