Homework 24: Cylindrical/Spherical integration

This homework is due Monday, 11/11.

1 Evaluate the following integrals.

a) $\int_0^{\pi} \int_0^{2} \int_0^{9-r^2} 7r \, dz \, dr \, d\theta$

b) $\int_0^{2\pi} \int_{\pi/2}^{\pi} \int_1^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

Solution:

a) The region of integration is given in cylindrical coordinates by $E = \{(r, \theta, z) \mid 0 \le \theta \le \pi\,, 0 \le r \le 2\,, 0 \le z \le 9 - r^2\}$. This represents the solid region in the first and second octants enclosed by the circular cylinder $r = 2$, bounded above by $z = 9 - r^2$, a circular paraboloid, and bounded below by the xy-plane. To evaluate:

$$\begin{aligned} 7\int_0^{\pi}\int_0^{2}\int_0^{9-r^2} r\,dz\,dr\,d\theta &= 7\int_0^{\pi}\int_0^{2}\left[rz\right]_{z=0}^{z=9-r^2}\,dr\,d\theta \\ &= 7\int_0^{\pi}\int_0^{2} r(9-r^2)\,dr\,d\theta \\ &= 7\int_0^{\pi} d\theta\,\int_0^{2}(9r - r^3)\,dr \\ &= 7\left[\theta\right]_0^{\pi}\left[\frac{9}{2}r^2 - \frac{1}{4}r^4\right]_0^2 \\ &= 7\pi(18-4) = 98\pi \end{aligned}$$

b) The region of integration is given in spherical coordinates by $E = \{(\rho, \theta, \phi) \mid 1 \le \rho \le 2\,, 0 \le \theta \le 2\pi\,, \pi/2 \le \phi \le \pi\}$. This represents the solid region between the spheres $\rho = 1$ and $\rho = 2$ and below the xy-plane.

$$\begin{aligned} \int_0^{2\pi}\int_{\pi/2}^{\pi}\int_1^{2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta &= \int_0^{2\pi} d\theta\,\int_{\pi/2}^{\pi}\sin\phi\,d\phi\,\int_1^2\rho^2\,d\rho \\ &= \left[\theta\right]_0^{2\pi}\left[-\cos\phi\right]_{\pi/2}^{\pi}\left[\frac{1}{3}\rho^3\right]_1^2 \\ &= 2\pi\,(1)\left(\frac{7}{3}\right) = \frac{14\pi}{3} \end{aligned}$$

2 Use cylindrical coordinates to find the volume of the solid that lies within both the cylinder $x^2+y^2 = 1$ and the sphere $x^2+y^2+z^2 = 4$.

Solution:

In cylindrical coordinates E is the solid region within the cylinder $r = 1$ bounded above and below by the sphere $r^2 + z^2 = 4$, so $E = \{(r, \theta, z) \mid 0 \le \theta \le 2\pi\,,\ 0 \le r \le 1\,, -\sqrt{4 - r^2} \le z \le \sqrt{4 - r^2}\}$. Thus the volume is

$$\begin{aligned}
\iiint_E dV &= \int_0^{2\pi} \int_0^1 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi} \int_0^1 2r\sqrt{4 - r^2}\,dr\,d\theta \\
&= \int_0^{2\pi} d\theta \int_0^1 2r\sqrt{4 - r^2}\,dr \\
&= 2\pi \left[-\frac{2}{3}(4 - r^2)^{3/2} \right]_0^1 \\
&= \frac{4}{3}\pi(8 - 3^{3/2})
\end{aligned}$$

3 Use spherical coordinates to evaluate

$$\int\int\int_H (16 - x^2 - y^2)\,dV\ ,$$

where H is the solid hemisphere $x^2 + y^2 + z^2 \le 16$, $z \ge 0$.

Solution:

In spherical coordinates, $H = \{(\rho, \theta, \phi) \mid 0 \leq \rho \leq 4\,,\, 0 \leq \theta \leq 2\pi\,,\, 0 \leq \phi \leq \pi/2\}$. Thus

$$\iiint_H \left(16 - x^2 - y^2\right)\, dV$$

$$\begin{aligned}
&= \int_0^{\pi/2} \int_0^{2\pi} \int_0^4 \left[16 - \left(\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta\right)\right] \rho^2 \sin\phi d\rho d\theta d\phi \\
&= \int_0^{\pi/2} \int_0^{2\pi} \int_0^4 \left(16 - \rho^2 \sin^2\phi\right) \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi \\
&= \int_0^{\pi/2} \int_0^{2\pi} \left[\frac{16\rho^3}{3} - \frac{\rho^5}{5}\sin^2\phi\right]_{\rho=0}^{\rho=4} \sin\phi\, d\theta\, d\phi \\
&= \int_0^{\pi/2} \int_0^{2\pi} \left(\frac{1024}{3}\sin\phi - \frac{1024}{5}\sin^3\phi\right) d\theta\, d\phi \\
&= \int_0^{2\pi} d\theta \int_0^{\pi/2} \left[-\frac{1024}{3}\cos\phi - \frac{1024}{5}\left(1 - \cos^2\phi\right)\sin\phi\right] \\
&= 2\pi \left[-\frac{1024}{3}\cos\phi - \frac{1024}{5}\left(\frac{\cos^3\phi}{3} - \cos\phi\right)\right]_0^{\pi/2} \\
&= 2\pi \left[0 + \frac{1024}{3} + \frac{1024}{5}\left(-\frac{2}{3}\right)\right] \\
&= \frac{2048}{5}\pi
\end{aligned}$$

4 Evaluate the integral by changing to spherical coordinates.

$$\int_{-a}^{a} \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} \left(x^2 z + y^2 z + z^3\right) dz\, dx\, dy$$

Solution:
The region of integration is the solid sphere $x^2+y^2+z^2 \le a^2$, so $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$, and $0 \le \rho \le a$. Also $x^2z+y^2z+z^3 = (x^2+y^2+z^2)z = \rho^2 z = \rho^3\cos\phi$, so the integral becomes

$$\int_0^\pi \int_0^{2\pi} \int_0^a (\rho^3 \cos\phi)\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$\begin{aligned}
&= \int_0^\pi \sin\phi\cos\phi \, d\phi \int_0^{2\pi} d\theta \int_0^a \rho^5 \, d\rho \\
&= \left[\frac{1}{2}\sin^2\phi\right]_0^\pi [\theta]_0^{2\pi} \left[\frac{1}{6}\rho^6\right]_0^a \\
&= 0
\end{aligned}$$

5 Show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{x^2+y^2+z^2}\, e^{-(x^2+y^2+z^2)} \, dx \, dy \, dz = 2\pi \ .$$

Solution:

The integral is

$$\lim_{R\to\infty}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\rho e^{-\rho^2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

We can separate out the θ and ϕ components. To integrate $\rho^3 e^{-\rho^2}$, we have to use integration by parts. Let $u=\rho^2$ and $dv=\rho e^{-\rho^2}d\rho$. We compute:

$$\lim_{R\to\infty}\int_0^{2\pi}\,d\theta\,\int_0^{\pi}\sin\phi\,d\phi\,\int_0^{R}\rho^3 e^{-\rho^2}d\rho$$

$$\begin{aligned}
&= 2\pi(2)\left(\left[\rho^2\left(-\frac{1}{2}\right)e^{-\rho^2}\right]_0^R-\int_0^R 2\rho\left(-\frac{1}{2}\right)e^{-\rho^2}\,d\rho\right)\\
&= \lim_{R\to\infty}4\pi\left(-\frac{1}{2}R^2e^{-R^2}+\left[-\frac{1}{2}e^{-\rho^2}\right]_0^R\right)\\
&= 4\pi\lim_{R\to\infty}\left[-\frac{1}{2}R^2e^{-R^2}-\frac{1}{2}e^{-R^2}+\frac{1}{2}\right]\\
&= 4\pi\left(\frac{1}{2}\right)=2\pi
\end{aligned}$$

We have used that $R^2e^{-R^2}\to 0$ as $R\to\infty$.

Main definitions

The integration factor in cylindrical coordinates $(x, y, z) = (r\cos(\theta), r\sin(\theta), z)$ is r as in polar coordinates.

The integration factor in spherical coordinates $(x, y, z) = (\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))$ is $\rho^2\sin(\phi)$. This was the surface area element $|\vec{r}_\phi \times \vec{r}_\theta|$

To evaluate an integral in spherical coordinates, we express the region in spherical coordinates, substitute $x = \rho\sin(\phi)\cos(\theta)$, $y = \rho\sin(\phi)\sin(\theta)$ and $z = \rho\cos(\phi))$ in the function and include the integration factor $\rho^2\sin(\phi)$.