Homework 23: Triple integrals

This homework is due Friday, 11/8.

1 a) Evaluate the iterated integral

$$\int_0^1 \int_0^z \int_0^{y^2} x^4 \, dx \, dy \, dz \ .$$

b) Which of the three following scrambled versions make sense too? (you don't have to evaluate)

$$\int_0^2 \int_0^{y^3} \int_0^z x^2 \, dx \, dz \, dy \ .$$

$$\int_0^z \int_0^2 \int_0^{x^3} x^2 \, dx \, dz \, dx \ .$$

$$\int_0^2 \int_0^y \int_0^{z^3} x^2 \, dx \, dz \, dy \ .$$

2 Evaluate the triple integral $\iiint_E 10yz \cos(x^5) \, dV$ over the solid given by

$$E = \{(x, y, z) \mid 0 \leq x \leq \pi/2, \ 0 \leq y \leq x, \ x \leq z \leq 2x\} \ .$$

3 Evaluate the triple integral

$$\int \int \int_E xy \, dV \ ,$$

where E is bounded by the parabolic cylinders $y = 3x^2$ and $x = 3y^2$ and the planes $z = 0$ and $z = x + y$.

4 Use a triple integral to find the volume of the given solid enclosed by the paraboloid $x = y^2 + z^2$ and the plane $x = 25$.

5 Find the moment of inertia

$$I = \int \int \int_E (x^2 + y^2) \ dzdxdy$$

about the z-axis of the solid $E : 0 \leq r \leq z \leq 16$.

Main definitions

If $f(x,y,z)$ is a function and E is a **solid**, then $\int\int\int_E f(x,y,z)\ dV$ is defined as the $n \to \infty$ limit of the Riemann sum

$$\frac{1}{n^3} \sum_{(\frac{i}{n},\frac{j}{n},\frac{k}{n})\in E} f(\frac{i}{n},\frac{j}{n},\frac{k}{n}) .$$

Like $dA = dxdy$ was a symbol for a small area, $dV = dxdydz$ indicates a small volume. Triple integrals are solved as a nested list of single integrals.

If $f(x,y,z) = 1$ then $\int\int\int_E 1\ dxdydz$ is the volume of the solid

A common situation is where the triple integral is reduced to a double integral

$$\int\int_R [\int_{g(x,y)}^{h(x,y)} f(x,y,z)\ dz]\ dxdy .$$

This is by far the most common case. For example, if $g(x,y) = 0$ and $f(x,y,z) = 1$, then

$$\int\int_R [\int_0^{h(x,y)} 1\ dz]\ dxdy = \int\int_R h(x,y)\ dxdy$$

is the signed volume of the solid under the graph of h. In variable calculus, where you were sometimes able to compute triple integrals by reducing to a single integral. You would have written $\int_a^b A(z)\ dz$ to compute the volume of a solid sandwiched between $z = a$ and $z = b$ for which the area of the cross section at height z is $A(z)$. In multi variable calculus we are much more flexible as we can now also reduce to a double integral.