Homework 22: Surface area

This homework is due Monday, 11/4/2019.

1 Find the area of the surface given by the **helicoid**

$$\vec{r}(u,v) = [u\cos(v), u\sin(v), v] \ .$$

with $0 \le u \le 1, \ 0 \le v \le \pi$.

Solution:
We compute:

$$\vec{r}_u = [\cos v, \sin v, 0] \qquad \vec{r}_v = [-u\sin v, u\cos v, 1] \ .$$

Therefore, $\vec{r}_u \times \vec{r}_v = [\sin v, -\cos v, u]$. The area is given by

$$\int_0^\pi \int_0^1 \sqrt{1+u^2} du dv = \frac{\pi}{2}(\sqrt{2} + \log(1+\sqrt{2}))$$

2 Find the surface area of the surface given as the graph

$$z = \frac{2}{3}(x^{3/2} + y^{3/2}), \ 0 \le x \le 3, \ 0 \le y \le 3 \ .$$

Solution:
We compute: $f_x = x^{1/2}$ and $f_y = y^{1/2}$. The surface area is

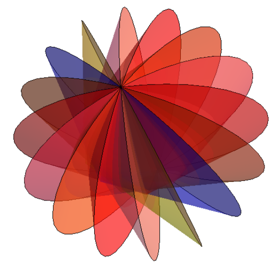
$$\int_0^3 \int_0^3 \sqrt{1+x+y} \cdot dx dy = \frac{2}{3} \int_0^3 \{(4+y)^{3/2} - (4+y)^{3/2}\} dy$$

This evaluates to $28/15(7\sqrt{7} - 9)$.

3 A decorative paper lantern is made of 8 surfaces. Each is parametrized by

$$\vec{r}(t,z) = [10z\cos(t), 10z\sin(t), z]$$

with $0 \leq t \leq 2\pi$ and $0 \leq z \leq 1$ and then translated or rotated. Find the total surface area of the lantern.



Solution:
We compute

$$\begin{aligned}\vec{r}_t &= [-10z\sin(t), 10z\cos(t), 0]\\ \vec{r}_z &= [10\cos(t), 10\sin(t), 1]\\ \vec{r}_t \times \vec{r}_z &= [10z\cos(t), 10z\sin(t), -100z]\end{aligned}$$

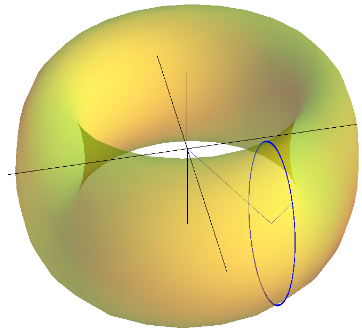
and the length is $|\vec{r}_t \times \vec{r}_z| = 10z\sqrt{101}$.

$$\int_0^{2\pi}\int_0^1 10z\sqrt{101}\,dzdt = 10\pi\sqrt{101}\ .$$

There are 8 pieces so that the final result is $\boxed{80\pi\sqrt{101}}$.

4 The figure shows the torus obtained by rotating about the z-axis the circle in the xz-plane with center $(5,0,0)$ and radius 1. Parametric equations for the torus are

$$\begin{aligned}x &= 5\cos\theta + \cos\alpha\cos\theta\\ y &= 5\sin\theta + \cos\alpha\sin\theta\\ z &= \sin\alpha\ ,\end{aligned}$$

where θ and α are the angles shown in the figure. Find the surface area of the torus. 

Solution:
Let $\vec{r} = [x, y, z]$. Then,

$$\vec{r}_\alpha = [-\sin\alpha\cos\theta, -\sin\alpha\sin\theta, \cos\alpha]$$

and

$$\vec{r}_\theta = [-5\sin\theta - \cos\alpha\sin\theta, 5\cos\theta + \cos\alpha\cos\theta, 0] \, .$$

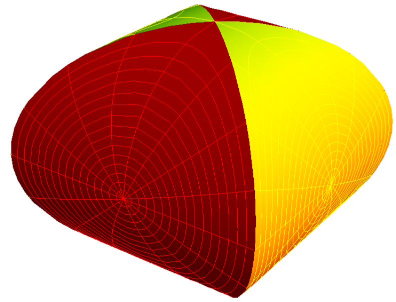
Thus

$$\vec{r}_\alpha \times \vec{r}_\theta = [-5\cos\alpha\cos\theta - \cos^2\alpha\cos\theta,$$
$$-5\cos\alpha\sin\theta - \cos^2\alpha\sin\theta, -5\sin\alpha - \cos\alpha\sin\alpha].$$

Therefore $|\vec{r}_\alpha \times \vec{r}_\theta| = (5 + \cos\alpha)$ and so the surface area is

$$\int_0^{2\pi}\int_0^{2\pi}(5 + \cos\alpha)d\alpha d\theta = 2\pi \cdot 2 \cdot 5\pi = 20\pi^2$$

5 The volume and surface area of the solid obtained by intersecting the solid cylinder $y^2+z^2 \leq 4$ with the solid cylinder $x^2+z^2 \leq 4$ has been found by Archimedes. Find the surface area of the surface

S bounding this solid. 

Solution:
In the picture above, the boundary of the solid is made up of two red and two yellow pieces. The yellow piece facing up lies on the cylinder $x^2 + z^2$ between $|x| \geq |y|$. This piece is the graph of a function $f(x, y) = \sqrt{4 - x^2}$. Computing $|r_x \times r_y|$ simplifies to $2/\sqrt{4 - x^2}$. The surface area of 1/8'th is

$$\int_0^2 \int_{-x}^{x} |r_x \times r_y| \, dydx = \int_0^2 2x \frac{2}{\sqrt{4 - x^2}} \, dx = 8 \; .$$

The answer is $8 \times 8 = 64$.
P.S. We can also use polar coordinates but that is a bit more complicated. The surface area of *one* piece is

$$\int_{-2}^{2} \int_{-\arccos|y|}^{\arccos|y|} d\theta dy = \int_{-2}^{2} 2 \arccos|y| dy = 8.$$

Again the surface area is $8 \cdot 8 = 64$.

Main definitions:

A surface $\vec{r}(u, v)$ parametrized on a parameter domain R has the **surface area**

$$\int\int_R |\vec{r}_u(u, v) \times \vec{r}_v(u, v)| \; dudv \; .$$

Examples:

$\vec{r}(u, v)$	$\lvert\vec{r}_u \times \vec{r}_v\rvert$
$[\rho\cos(u)\sin(v), \rho\sin(u)\sin(v), \rho\cos(v)]$	$\rho^2 \lvert\sin(v)\rvert$
$[u, v, f(u, v)]$	$\sqrt{1 + f_u^2 + f_v^2}$
$[f(v)\cos(u), f(v)\sin(u), v]$	$f(v)\sqrt{1 + f'(v)^2}$