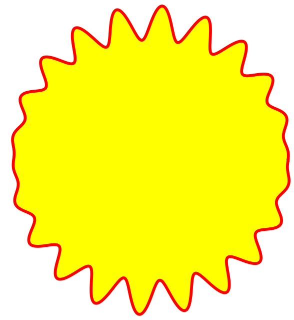
Homework 21: Polar integration

This homework is due Friday, 11/1/2019.

1 Find the area of the **pancake region**

$$\iint_R 1 \; dA \; ,$$

where R is given in polar coordinates as $0 \leq r \leq 8 - \sin(\theta)\cos(21\theta)$.



Solution:
$\int_0^{2\pi} (8 - \sin(t)\cos(21t))^2/2 \; d\theta = 257\pi/4 = (64 + 1/4)\pi$.

2 Evaluate the following integral by changing to polar coordinates:

$$\iint_R 17x \; dA \; ,$$

where R is the region in the first quadrant that lies between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 2x$.

Solution:
In polar coordinates, the first equation of the first circle is $r = 2$ while the second circle is $r^2 = 2r\cos\theta$ or $r = 2\cos\theta$. Using the fact that $2\cos\theta < 2$, the integral over D is given by

$$17\int_0^{\pi/2} d\theta \int_{2\cos\theta}^{2} (r\cos\theta) r dr = 17\int_0^{\pi/2} \cos\theta d\theta \cdot \frac{8 - 8\cos^3\theta}{3} .$$

Using that the integral $\int_0^{\pi/2} \cos\theta d\theta = 1$ and $\int_0^{\pi/2} \cos^4\theta d\theta = 3\pi/16$, we see that

$$\iint_D 17x dA = 17\frac{8}{3} - \frac{8}{3} \cdot \frac{3\pi}{16} = 17(8/3 - \pi/2) = 18.6298... .$$

3 Use polar coordinates to find the volume of the solid bounded by the paraboloids $z = 3x^2 + 3y^2$ and $z = 4 - x^2 - y^2$.

Solution:
In polar coordinates, the equations of the paraboloids become $z = 3r^2$ and $z = 4 - r^2$. Since the first paraboloid is convex, while the second one is concave, the solid in question is above the first paraboloid and below the second. The two meet in the circle given by the equations $r = 1, z = 1$. Therefore, we must integrate

$$\int_0^{2\pi} d\theta \int_0^1 r dr \int_{3r^2}^{4-r^2} 1 dz = 2\pi \int_0^1 r(4 - 4r^2) = 2\pi(4 - 3).$$

This simplifies to 2π.

4 a) A city near the sea is modeled by a half disk $D = \{(x, y) \mid x^2 + y^2 \leq 49, x \geq 0\}$. with center the origin and radius 7. What is the average distance of a point in D to the origin? In other words,

what is the integral

$$\frac{\iint_D \sqrt{x^2+y^2}\;dxdy}{\iint_D 1\;dxdy} \; .$$

b) The distance to the beach is x. Find the average distance

$$\iint_D x\;dxdy / \iint_D 1\;dxdy$$

to the beach.

Solution:
a) We work in polar coordinates and $a = 7$.

$$\frac{1}{\pi a^2} \int_{-\pi/2}^{\pi/2} \int_0^a r \cdot r dr d\theta = \frac{1}{2\pi a^2} \cdot \pi \cdot a^3/3 = 2a/3 = 14/3 \; .$$

b) $\int_0^a \int_{-\pi/2}^{\pi/2} r\cos(\theta) r \; dr d\theta / (\pi a^2/2) = 28/(3\pi)$.

5 Evaluate the iterated integral

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} 9\sqrt{x^2+y^2}\;dy\,dx \; .$$

Solution:
The integrand is $9\sqrt{x^2+y^2} = 9r$. The region in question is a semicircle centered at 1 with radius 1. If $x = r\cos\theta$ and $y = r\sin\theta$, then $y = \sqrt{2x-x^2}$ simplifies to $r = 2\cos\theta$. Thus the integral is

$$\int_0^{\pi/2} \int_0^{2\cos\theta} 9r \cdot r dr d\theta = \int_0^{\pi/2} 24 \cdot \cos^3\theta d\theta = 16.$$

Main definitions

Polar coordinates $(x, y) = (r\cos(t), r\sin(t))$ allow to describe regions bound by polar curves $(r(\theta), \theta)$.

The **average** of a quantity $f(x, y)$ over a region G is the fraction

$$\frac{\iint_G f(x,y)\; dA}{\iint_G 1\; dA} .$$

To integrate in polar coordinates, we evaluate the integral

$$\iint_R f(x,y)\; dxdy = \iint_R f(r\cos(\theta), r\sin(\theta))r\; drd\theta \ ,$$

where R is described in polar coordinates.

Example:

To integrate $f(x, y) = x^2 + y^2$ over the region $x^2 + y^2 \leq 9, x \geq 0, y \geq 0$, we integrate

$$\int_0^{\pi/2} \int_0^3 r^2 \cdot r\; drd\theta = (\pi/2)(3^4/4) = 81\pi/8 .$$