Homework 20: Double integrals

This homework is due Wednesday, 10/30/2019.

1 a) We use the notation $\log(x) = \ln(x)$. Compute

$$\int_1^2 \int_2^4 \frac{\log(x)}{xy^3} \, dy \, dx \, .$$

b) Now get

$$\int_1^2 \int_2^4 \frac{\log(x)}{xy^3} \, dx \, dy \, .$$

c) Why is the fact that a) and b) give different results not provide a counter example to Fubini's theorem?

2 For a function $f(x)$ which is a probability density, meaning $f \geq 0, \int_R f(x)dx = 1$, the integral

$$E = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |x - y| f(x) f(y) dx dy$$

is called the Gini energy. The Gini index E/m, where $m = \int_{-\infty}^{\infty} x f(x) dx$ is the **mean**, plays a role in computing wealth inequality.
a) Find the Gini index for $f(x) = 1$ on $[0, 1]$ and $f(x) = 0$ else.
b) Find the Gini energy for the normal distribution $f(x) = \exp(-x^2)/\sqrt{\pi}$. (Use technology to get a numerical answer.)

3 We write dA if we don't want specify the order of integration yet. Evaluate

$$\iint_R \log(1 + x)\sqrt{x/y} \, dA$$

with $\log(x) = \ln(x)$ over the region $R = \{(x, y) \mid 0 \leq y \leq x, 0 \leq x \leq 1 \}$.

4 a) Evaluate $\int_0^2 \int_y^{2y} 6xy \, dx \, dy$.

b) Compute the same integral as a $dydx$ integral. You might have to split the integral up into two integrals.

5 Evaluate $\int_0^8 \int_{\sqrt[3]{y}}^2 4e^{(x^4)} dx \, dy$.

Main definitions

If R is a planar region and $f(x, y)$ a function of two variables, the **double integral** $\int \int_R f(x, y) \ \ dA$ is the limit of the Riemann sum $(1/n^2) \sum_{(i/n, j/n) \in R} f(i/n, j/n)$ for $n \to \infty$. A **dydx-region** is of the form

$$R = \{(x, y) \mid a \leq x \leq b, \ c(x) \leq y \leq d(x) \ \} \ .$$

This leads to a *dydx*-**integral**

$$\iint_R f \ dA = \int_a^b \int_{c(x)}^{d(x)} f(x, y) \ dydx \ .$$

A **dxdy-region** is of the form

$$R = \{(x, y) \mid c \leq y \leq d, a(y) \leq x \leq b(y) \ \} \ .$$

This leads to a *dxdy*-**integral**

$$\iint_R f \ dA = \int_c^d \int_{a(y)}^{b(y)} f(x, y) \ dxdy \ .$$

Fubini's theorem allows to switch the order of integration over a rectangle, if the function f is continuous:

$$\int_a^b \int_c^d f(x, y) \ dxdy = \int_c^d \int_a^b f(x, y) \ dydx \ .$$