# Homework 20: Double integrals

This homework is due Wednesday, 10/30/2019.

1 a) We use the notation $\log(x) = \ln(x)$. Compute

$$\int_1^2 \int_2^4 \frac{\log(x)}{xy^3} \, dy \, dx \ .$$

b) Now get

$$\int_1^2 \int_2^4 \frac{\log(x)}{xy^3} \, dx \, dy \ .$$

c) Why is the fact that a) and b) give different results not provide a counter example to Fubini's theorem?

**Solution:**
a) The first integral is $3\log(2)^2/64$.
b) The second integral is $9\log(2)^2/16$.
c) One would also have to change the bounds too.

2 For a function $f(x)$ which is a probability density, meaning $f \geq 0$, $\int_R f(x)dx = 1$, the integral

$$E = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |x-y| f(x) f(y) dx dy$$

is called the Gini energy. The Gini index $E/m$, where $m = \int_{-\infty}^{\infty} x f(x) dx$ is the **mean**, plays a role in computing wealth inequality.
a) Find the Gini index for $f(x) = 1$ on $[0, 1]$ and $f(x) = 0$ else.
b) Find the Gini energy for the normal distribution $f(x) = \exp(-x^2)/\sqrt{\pi}$. (Use technology to get a numerical answer.)

**Solution:**
a) $2/3$
b) $\sqrt{2/\pi}$.

3 We write $dA$ if we don't want specify the order of integration yet. Evaluate

$$\iint_R \log(1+x)\sqrt{x/y}\,dA$$

with $\log(x) = \ln(x)$ over the region $R = \{(x,y) \mid 0 \leq y \leq x, 0 \leq x \leq 1\}$.

**Solution:**
Integrate first over $y$. Note that $\log(1+x)$ does not depend on $y$. Then integrate over $x$. The answer is $1/2$.

4 a) Evaluate $\int_0^2 \int_y^{2y} 6xy\,dx\,dy$ .
b) Compute the same integral as a $dydx$ integral. You might have to split the integral up into two integrals.

**Solution:**

a)

$$\begin{aligned}\int_0^2 \int_y^{2y} 6xy\,dx\,dy &= \int_0^2 \frac{1}{2} 6x^2y\Big|_{x=y}^{x=2y}\,dy \\ &= \frac{1}{2}\int_0^2 6y(4y^2-y^2)\,dy = \frac{1}{2}\int_0^2 27y^3\,dy \\ &= \frac{3}{2}(\frac{1}{4}6y^4)\Big|_0^2 = 6(4-0) = 36.\end{aligned}$$

b) Make a picture to get the region. Then compute

$$\int_0^2 \int_{x/2}^{x} xy\ dydx + \int_2^4 \int_{x/2}^{2} xy\ dydx\ .$$

This is $9+27=36$ again.

5 Evaluate $\int_0^8 \int_{\sqrt[3]{y}}^2 4e^{(x^4)} dx\,dy$.

**Solution:**

$$\begin{aligned}4\int_0^8 \int_{\sqrt[3]{y}}^2 e^{(x^4)}\,dx\,dy &= 4\int_0^2 \int_0^{x^3} e^{(x^4)}\,dy\,dx \\ &= 4\int_0^2 e^{(x^4)}[y]_{y=0}^{y=x^3}\,dx = 4\int_0^2 x^3e^{x^4}\,dx \\ &= e^{x^4}]_0^2 = (e^{16}-1)\end{aligned}$$

## Main definitions

If $R$ is a planar region and $f(x,y)$ a function of two variables, the **double integral** $\int\int_R f(x,y) \ dA$ is the limit of the Riemann sum $(1/n^2)\sum_{(i/n,j/n)\in R} f(i/n,j/n)$ for $n \to \infty$. A **dydx-region** is of the form

$$R = \{(x,y) \mid a \leq x \leq b, \ c(x) \leq y \leq d(x) \} .$$

This leads to a *dydx*-**integral**

$$\iint_R f \ dA = \int_a^b \int_{c(x)}^{d(x)} f(x,y) \ dydx .$$

A **dxdy-region** is of the form

$$R = \{(x,y) \mid c \leq y \leq d, a(y) \leq x \leq b(y) \} .$$

This leads to a *dxdy*-**integral**

$$\iint_R f \ dA = \int_c^d \int_{a(y)}^{b(y)} f(x,y) \ dxdy .$$

**Fubini's theorem** allows to switch the order of integration over a rectangle, if the function $f$ is continuous:

$$\int_a^b \int_c^d f(x,y) \ dxdy = \int_c^d \int_a^b f(x,y) \ dydx .$$