Homework 19: Global extrema

This homework is due Monday, 10/28/2019.

1 a) We suppose that the Cobb Douglas production formula $Q(L,K) = L^{1/4}K^{3/4} = 100$, which tells that the quantity Q is constant. What values of L and K minimizes the cost function $C(L,K) = 4L + 5K$ under the constraint $Q(L,K) = 100$?
b) Is there a global maximum or minimum for $C(L,K)$ on the region $L \geq 0, K \geq 0$ without the constraint $Q = 100$? If yes, what is the maximum, or what is the minimum?

Solution:
$C(L,K) = 4L + 5K$, $g(L,K) = L^{1/4}K^{3/4} = 100 \Rightarrow \nabla C = [4,5]$, $\lambda\nabla g = [\lambda\frac{1}{4}bL^{-3/4}K^{3/4}, \lambda\frac{3}{4}L^{1/4}K^{-1/4}]$. Then $\dfrac{4}{1/4}(\dfrac{L}{K})^{3/4} = \dfrac{5}{3/4}(\dfrac{K}{L})^{1/4}$ and $L^{1/4}K^{3/4} = 100 \Rightarrow \frac{5/4}{4(3/4)} = (\frac{L}{K})^{3/4}(\frac{L}{K})^{1/4} \Rightarrow L = \frac{K(5/4)}{4(3/4)}$. This gives $K = (12/5)L$. Plugging this into the production formula, we get $L = [\dfrac{K(5/4)}{4(3/4)}]^{1/4}K^{3/4} = 100$. Hence $K = \dfrac{100(4)^{1/4}(3/4)^{1/4}}{(5)^{1/4}(1/4)^{1/4}} = \dfrac{100}{1}(\dfrac{12}{5})^{1/4}$ and $L = \dfrac{100(5)^{3/4}(1/4)^{3/4}}{4^{3/4}(3/4)^{3/4}} = \dfrac{100}{1}(\dfrac{5}{12})^{3/4}$. There is a minimum 0 at $L = 0$ or $K = 0$. There is no global maximum.

2 The extremal value theorem assures that on $D = \{x^2+y^2 \leq 1\}$, a continuous function (and especially a differentiable function) has both a at least one maximum and at least one minimum on D. On an open disk essentially anything goes, as you can see here:

a) Find a differentiable $f(x,y)$ which has exactly one maximum and no minimum on $x^2+y^2<1$. b) Engineer a differentiable function $f(x,y)$ which has exactly one maximum and exactly one minimum on $x^2+y^2<1$ and no saddle point and no other critical point. c) Engineer a differentiable function $f(x,y)$ which has exactly two maxima and no other critical point on $x^2+y^2<1$.

Solution:
a) $f(x,y) = 1 - x^2 - y^2$ does the job
b) $f(x,y) = (x^3 - x)(1 - y^2)$ works. One strategy to look for a point is to get a case with a maximum and minimum and then modify the x and y so that inside the unit circle, one has just one max and one min.
c) Here is an idea: take a two hill function so that the hills are at $x = a, -a$ for soma $a < 1$ Then multiply with x so that the y axes has the same height. Then tilt the graph a bit. A nice example has been given by John Cook (https://www.johndcook.com/blog/2017/10/04/no-critical-point-between-two-peaks) $f(x,y) = -(x^2-1)^2-(x^2y-x-1)^2$. Two years ago, students came up with many solutions $http://www.math.harvard.edu/archive/21a\_fall\_17/exhibits/challeng$

3 Find the absolute maximum and minimum values of

$$f(x,y) = e^{-x^2-y^2}(x^2+2y^2) \quad ,$$

on the disk $D = \{x^2+y^2 \le 4\}$.

Solution:
Inside D: $f_x = 2xe^{-x^2-y^2}(1 - x^2 - 2y^2) = 0$ implies $x = 0$ or $x^2 + 2y^2 = 1$. Then if $x = 0$, $f_y = 2ye^{-x^2-y^2}(2 - x^2 - 2y^2) = 0$ implies $y = 0$ or $2 - 2y^2 = 0$ giving the critical points $(0, 0), (0, \pm 1)$. If $x^2 + 2y^2 = 1$, then $f_y = 0$ implies $y = 0$ giving the critical points $(\pm 1, 0)$. Now $f(0, 0) = 0, f(\pm 1, 0) = e^{-1}$ and $F(0, \pm 1) = 2e^{-1}$. On the boundary of D: $x^2 + y^2 = 4$, so $f(x, y) = e^{-4}(4 + y^2)$ and f is smallest when $y = 0$ and largest when $y^2 = 4$. But $f(\pm 2, 0) = 4e^{-4}$, $f(0, \pm 2) = 8e^{-4} = 0.146....$ Thus on D the global maximum of f is $f(0, \pm 1) = 2e^{-1} = 0.73..$ and the global minimum is $f(0, 0) = 0$.

4 a) Use Lagrange multipliers to find the maximum and minimum values of f subject to the given constraint

$$f(x, y) = \frac{1}{x} + \frac{1}{y};\ g(x, y) = \frac{1}{x^2} + \frac{1}{y^2} = 1\ .$$

b) Is there a global maximum or global minimum of f on $g = 1$? Is this a case for the Bolzano theorem?

Solution:

a) $f(x,y) = \frac{1}{x} + \frac{1}{y}$, $g(x,y) = \frac{1}{x^2} + \frac{1}{y^2} = 1 \Rightarrow \nabla f = [-x^{-2}, -y^{-2}] = \lambda \nabla g = [-2\lambda x^{-3}, -2\lambda y^{-3}]$. Then $-x^{-2} = -2\lambda x^3$ or $x = 2\lambda$ and $-y^{-2} = -2\lambda y^{-3}$ or $y = 2\lambda$. Thus $x = y$, so $\frac{1}{x^2} + \frac{1}{y^2} = \frac{2}{x^2} = 1$ implies $x = \pm\sqrt{2}$ and the possible points are $(\pm\sqrt{2}, \pm\sqrt{2})$. The absolute maximum of f subject to $x^{-2} + y^{-2} = 1$ is then $f(\sqrt{2}, \sqrt{2}) = \sqrt{2}$ and the absolute minimum is $f(-\sqrt{2}, -\sqrt{2}) = -\sqrt{2}$.

b) There is a global minimum and maximum and you have found them in a). We could not apply Bolzano as the curve is not bounded. Why is there a global max and min? There is a global maximum on $(\sqrt{2}, \sqrt{2})$ and a global minimum on $(-\sqrt{2}, -\sqrt{2})$. There are four branches to the constraint $1/x^2 + 1/y^2 = 1$. In the first quadrant the function f is maximal at $(\sqrt{2}, \sqrt{2})$ and goes to 1 when either x goes to infinity or y goes to infinity. In the third quadrant, f is minimal at $(-\sqrt{2}, -\sqrt{2})$ and goes to -1 as x goes to infinity or minus infinity. On the second and third quadrant, the function is monotone going from -1 to 1.

5 A package in the shape of a rectangular box can be mailed by the **US Postal Service** if the sum of its length and girth (the perimeter of a cross-section perpendicular to the length) is at most 108. Find the dimensions of the package with largest volume $V(x,y,z) = xyz$ that can be mailed under the constraint $x + 2y + 2z \leq 108$.

Solution:
We want to extremize $V = xyz$ where $x + 2y + 2z \leq 108$, $x \geq 0$, $y \geq 0$, $z \geq 0$. First maximize V subject to $x+2y+2z = 108$ with x, y, z all positive. Then $[yz,\ xz,\ xy] = [\lambda, 2\lambda, 2\lambda]$ implies $2yz = xz$ or $x = 2y$ and $xz = xy$ or $z = y$. Thus $g(x, y, z) = 108$ implies $6y = 108$ or $y = 18 = z$, $x = 36$, so the volume is $V = 11,664$ cubic units. The maximal dimension is $(36, 18, 18)$. At the boundary , where $x = 0$ or $y = 0$ or $z = 0$, we get 0 as a volume.

Main definitions:

Standard assumption is still that all functions have continuous first and second derivatives. Maximum means local maximum etc. A point (x_0, y_0) is an **absolute maximum = global maximum** on a domain R, if $f(x, y) \leq f(x_0, y_0)$ for all (x, y) in R.

To find a global maximum, we look at the local maxima and minima as well as the maxima and minima on the boundary. The later is a Lagrange problem. If the domain is unbounded, we also have to look at the behavior of the function when $x, y \to \infty$.

Extremal value theorem of Bolzano: On a bounded closed region or bounded closed curve, there is always a global maximum and global minimum.