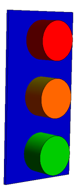
Homework 18: Lagrange multipliers

This homework is due Friday, 10/25. Always use the Lagrange method.

1 a) We look at a melon shaped candy. The outer radius is x, the inner is y. Assume we want to extremize the **sweetness function** $f(x,y) = -x^2+2y^2$ under the constraint that $g(x,y) = x-y = 2$. Since this problem is so tasty, we require you to use the most yummy method known to mankind: the **Lagrange** method! Is your solution a minimum or maximum?



b) The material to build a traffic light is $g(x,y) = 6 + 6\pi xy + 3\pi x^2 = 12$ is fixed (the radius of each cylinder is x and the height is y and the constant 6 is the material for the back plate). We want to build a light for which the shaded region with volume $f(x,y) = 3\pi x^2 y$ is maximal.

2 The method of Lagrange multipliers can also be used with more than two variables. The equations $\nabla f = \lambda \nabla g, g = c$ are the same. Let $f(x,y,z) = xyz$ be the volume of a box which is open on the top and $g(x,y,z) = xy + 2xz + 2yz$ the surface area. a) Maximize the volume if surface area $g = 12$ is fixed? and b) Minimize the surface area if the volume $f = 2$ is fixed.

3 The method of Lagrange multipliers can also be used with more than one constraint, a situation often occurring in applications. This is not covered in class but explored by you in this HW (see the box). Find the maximum and minimum f under the two constraints:

$$f(x,y,z) = 3x - y - 3z;$$

$$g(x,y,z) = x + y - z = 0, h(x,y,z) = x^2 + 2z^2 = 1 .$$

4 Use Lagrange multipliers to prove that the triangle with maximum area that has a given perimeter 2 is equilateral. Why does the Lagrange method not establish minima? *Hint:* Use Heron's formula $A = \sqrt{s(s-x)(s-y)(s-z)}$ with $s = 1$.

5 Which pyramid of height h over a square $[-a, a] \times [-a, a]$ with surface area is $4a\sqrt{h^2 + a^2} + 4a^2 = 4$ has maximal volume $V(h, a) = 4ha^2/3$? By using new variables (x, y) and multiplying V with a constant, we get to the equivalent problem to maximize $f(x, y) = yx^2$ over the constraint $g(x, y) = x\sqrt{y^2 + x^2} + x^2 = 1$.

Main definitions

The system of equations $\nabla f(x, y) = \lambda \nabla g(x, y), g(x, y) = 0$ for the three unknowns x, y, λ are the **Lagrange equations**. λ is a **Lagrange multiplier**. The **two constraint case** appears only here in homework and is not covered in section $\nabla f(x, y, z) = \lambda \nabla g(x, y, z), g(x, y, z) = 0$ are the **Lagrange equations** $\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z), \quad g(x, y, z) = 0, h(x, y, z) = 0$ are the **Lagrange equations** with two constraints. **Lagrange theorem:** Maxima or minima of f on the constraint $g = c$ are either solutions of the Lagrange equations or critical points of g.