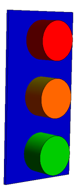
Homework 18: Lagrange multipliers

This homework is due Friday, 10/25. Always use the Lagrange method.

1 a) We look at a melon shaped candy. The outer radius is x, the inner is y. Assume we want to extremize the **sweetness function** $f(x,y) = -x^2+2y^2$ under the constraint that $g(x,y) = x-y = 2$. Since this problem is so tasty, we require you to use the most yummy method known to mankind: the **Lagrange** method! Is your solution a minimum or maximum?



b) The material to build a traffic light is $g(x,y) = 6 + 6\pi xy + 3\pi x^2 = 12$ is fixed (the radius of each cylinder is x and the height is y and the constant 6 is the material for the back plate). We want to build a light for which the shaded region with volume $f(x,y) = 3\pi x^2 y$ is maximal.

Solution:

a) The gradients of f and g are $\nabla f = [-2x, 2y], \nabla g = [1, -1]$. The Lagrange equations are $-2x = \lambda, 2y = -\lambda$. Elimination λ gives $x = 2y$. Plug this into the constraint. We get $\boxed{x = 4, y = 2}$ as the minimum.

b) The Lagrange equations $\nabla f = \lambda \nabla g, g = 12$ are

$$\begin{aligned} 6\pi xy &= \lambda(6\pi y + 6\pi x) \\ 3\pi x^2 &= \lambda(6\pi x) \\ 2\pi xy + \pi x^2 &= 6 \end{aligned}$$

Eliminating λ from the first two equations gives $x = y$. Plugging into the constraint gives $\boxed{x = \sqrt{2/(3\pi)} = y}$. The maximal value is $3\pi x^2 y = 2\sqrt{2/(3\pi)} = 0.921...$.

2 The method of Lagrange multipliers can also be used with more than two variables. The equations $\nabla f = \lambda \nabla g, g = c$ are the same. Let $f(x, y, z) = xyz$ be the volume of a box which is open on the top and $g(x, y, z) = xy + 2xz + 2yz$ the surface area. a) Maximize the volume if surface area $g = 12$ is fixed? and b) Minimize the surface area if the volume $f = 2$ is fixed.

Solution:

a) The Lagrange equations have the solution $(x, y, z) = (2, 2, 1)$.

b) The Lagrange equations look almost the same. It is just that the λ is on the other side. The solution is $(2^{2/3}, 2^{2/3}, 2^{-1/3})$. If we would have taken $f = 4$, then the critical points would have been the same.

3 The method of Lagrange multipliers can also be used with more

than one constraint, a situation often occurring in applications. This is not covered in class but explored by you in this HW (see the box). Find the maximum and minimum f under the two constraints:

$$f(x,y,z) = 3x - y - 3z;$$
$$g(x,y,z) = x + y - z = 0, h(x,y,z) = x^2 + 2z^2 = 1 \,.$$

Solution:
By the method of Lagrange multipliers,

$$[3,-1,-3] = \lambda[1,1,-1] + \mu[2x,0,4z]$$

The "middle" equation tells us that $\lambda = -1$. Hence, $2x\mu - 1 = 3$ and $4z\mu + 1 = -3$. In particular, $x\mu = 2$ and $-z\mu = 1$ so $\boxed{x = -2z}$. Plugging $x = -2z$ into the first constraint, we find $\boxed{y = 3z}$. Plugging $x = -2z$ in the second constraint, we see that $6z^2 = 1$. Thus the two critical points are $(\frac{2}{\sqrt{3}}, -\frac{3}{\sqrt{6}}, -\frac{1}{\sqrt{6}})$ and $(-\frac{2}{\sqrt{6}}, \frac{3}{\sqrt{6}}, \frac{1}{\sqrt{6}})$. The first gives the maximal value of $\frac{12}{\sqrt{6}}$ while the second gives the minimum value of $-\frac{12}{\sqrt{6}}$.

4 Use Lagrange multipliers to prove that the triangle with maximum area that has a given perimeter 2 is equilateral. Why does the Lagrange method not establish minima? *Hint:* Use Heron's formula $A = \sqrt{s(s-x)(s-y)(s-z)}$ with $s = 1$.

Solution:
a) To maximize the area A, it is sufficient to maximize A^2. We do so since the formula for A^2 does not involve a square root. If the perimeter p is fixed, then so is s. Hence, it is sufficient to maximize

$$\begin{aligned} f(x,y,z) &= s(s-x)(s-y)(s-z) \\ &= (y+z-x)(z+x-y)(x+y-z) \end{aligned}$$

provided $g(x,y,z) = x+y+z = 2s = 2$. By Lagrange multipliers,

$$\nabla f = \lambda[1,1,1].$$

Thus, $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z}$. We compute:

$$0 = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} = (x+y-z) \cdot -4(x-y)$$

By the triangle inequality, $x+y > z$, so $x = y$. Similarly, $x = z$ forcing the triangle to be equilateral.
b) The minimal area is obtained if either $x = s$ or $y = s$ or $z = s$ which corresponds to a degenerate triangle with zero area. At these minima, the function f is no more differentiable. Indeed, it is the boundary where the function is defined. The constraint $g = x+y+z = 2s$

5 Which pyramid of height h over a square $[-a,a]\times[-a,a]$ with surface area is $4a\sqrt{h^2+a^2}+4a^2 = 4$ has maximal volume $V(h,a) = 4ha^2/3$? By using new variables (x,y) and multiplying V with a constant, we get to the equivalent problem to maximize $f(x,y) = yx^2$ over the constraint $g(x,y) = x\sqrt{y^2+x^2}+x^2 = 1$.

Solution:

An elegant solution can be obtained by first simplifying the constraint and write $x^2(y^2 - x^2) - (1 - x^2)^2 = x^2y^2 + 2x^2 - 1 = 0$. Now the Lagrange equations are not so bad:

$$\begin{aligned} 2xy &= \lambda(4x + 2xy^2) \\ x^2 &= \lambda 2x^2 y \\ x^2y^2 + 2x^2 &= 1 \ . \end{aligned}$$

Eliminating λ by cross multiplying the first two equations gives $4x^3y^2 = x^2(4x + 2xy^2)$. Since $x = 0$ is not possible, we can divide both sides by x^3 and get $\boxed{y^2 = 2}$. Plugging into the constraint, we get $\boxed{x = 1/2}$. Without this simplification, the Lagrange system would have become more complicated:

$$\begin{aligned} 2xy &= \lambda(\sqrt{y^2 + x^2} + x^2/\sqrt{y^2 + x^2} + 2x) \\ x^2 &= \lambda yx/\sqrt{y^2 + x^2} \\ 1 &= x\sqrt{y^2 + x^2} + x^2 \end{aligned}$$

Main definitions

The system of equations $\nabla f(x,y) = \lambda \nabla g(x,y), g(x,y) = 0$ for the three unknowns x, y, λ are the **Lagrange equations**. λ is a **Lagrange multiplier**. The **two constraint case** appears only here in homework and is not covered in section $\nabla f(x,y,z) = \lambda \nabla g(x,y,z), g(x,y,z) = 0$ are the **Lagrange equations** $\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z)$, $g(x,y,z) = 0$, $h(x,y,z) = 0$ are the **Lagrange equations** with two constraints. **Lagrange theorem:** Maxima or minima of f on the constraint $g = c$ are either solutions of the Lagrange equations or critical points of g.