Homework 17: Extrema

This homework is due Wednesday, 10/23

1 Find the local maximum and minimum values of the function

$$f(x, y) = 8 + 4xy^2 + \frac{16}{x} + \frac{4}{y}\ .$$

Use the second derivative test to justify your answer.

Solution:
The gradient of f is $\nabla f(x, y) = [4y^2 - \frac{16}{x^2}, 8xy - \frac{4}{y^2}]$. It is solved by $(x, y) = (-4, -1/2)$ and $(x, y) = (4, 1/2)$. The second derivatives are $f_{xx} = 32/x^3$, $f_{xy} = 8y$ and $f_{yy} = 8x + 8/y^3$. The point $(-4, -1/2)$ is a maximum and $(4, 1/2)/3^{1/4}$ is a minimum.

2 Classify the critical points of the function

$$f(x, y) = 9e^{2y}(4y^2 - x^2)$$

using the second derivative test.

Solution:

Again, we compute the gradient

$$\nabla f = \left[9e^{2y}(-2x), 9e^{2y}(8y + 8y^2 - 2x^2)\right].$$

This is $(0, 0)$ if $-2x = 8y + 8y^2 - 2x^2 = 0$ so $x = 0$ and $y = 0, -1$. Thus, we must investigate the critical point $(0, 0)$ and $(0, -1)$. The matrix of second derivatives is 9

$$\begin{pmatrix} -2e^{2y} & -4xe^{2y} \\ -4xe^{2y} & e^{2y}(8 + 32y + 16y^2 - 4x^2)) \end{pmatrix}$$

At $(0, 0)$, this is 9

$$\begin{pmatrix} -2 & 0 \\ 0 & 8 \end{pmatrix}$$

which has a negative determinant and thus is a $\boxed{\text{saddle}}$.

At $(0, -1)$, this is

$$\begin{pmatrix} -2e^{-2} & 0 \\ 0 & -8e^{-2} \end{pmatrix}$$

which has positive determinant but negative first term, so is a local $\boxed{\text{maximum}}$.

3 Find the local maximum and minimum values and saddle point(s) of the function.

$$f(x, y) = -12 \sin x \sin y, -\pi < x < \pi, -\pi < y < \pi$$

Solution:
The gradient $\nabla f = [12\cos x \sin y, 12 \sin x \cos y]$. Since $\cos t$ and $\sin t$ cannot be simultaneously be 0, the gradient is 0 if and only if $\cos x = \cos y = 0$ or $\sin x = \sin y = 0$. Thus, the critical points in the range provided are $(\pm\pi/2, \pm\pi/2)$ and $(0, 0)$. The matrix of second derivatives is 12

$$\begin{pmatrix} -2\sin x \sin y & 2\cos x \cos y \\ 2\cos x \cos y & -2\sin x \sin y \end{pmatrix}$$

At $(0, 0)$, this is

$$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

which has negative determinant, so it is a saddle. At $(\pi/2, \pi/2)$ and $(-\pi/2, -\pi/2)$, it is 12

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

which yields a local minimum. Finally, at $(\pi/2, -\pi/2)$ and $(-\pi/2, \pi/2)$, we get 12

$$\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

which gives a local maximum.

4 Companies like **Netflix** or **Hulu** track movie preferences. One can visualize preferences on parameter spaces which is the **intelligence - emotion** plane. Based on viewing habits, the service decides what you want to see. Your profile is a function $f(x, y)$. Maximizing this function allows the company to pick movies for you. a) Assume that your user profile is the function $f(x, y) = -2x^3 + 9x^2 - 12x - y^2$. Find and classify all the critical points and especially find the local maxima of f.
b) Use a computer algebra system to find how many complex critical points the function $f(x, y) = 4x + 3y + x^3 + y^3 - x^4y - x^2y^2 + xy$ has. Locate the real ones and tell whether they are maxima, minima or saddle points.

Solution:
a) The gradient of f is $\nabla f = [-6x^2 + 18x - 12, -2y]$. This is 0 if $-6x^2 + 18x - 12 = -2y = 0$. The first equation gives $x^2 - 3x + 2 = 0$ from which $x = 1$ or 2. The second equation reduces to $y = 0$. Thus we must investigate $(1, 0)$ and $(2, 0)$. The matrix of second derivatives is

$$\begin{pmatrix} -12x + 18 & 0 \\ 0 & -2 \end{pmatrix}$$

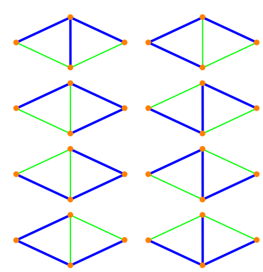
At the point $(1, 0)$, it is

$$\begin{pmatrix} 6 & 0 \\ 0 & -2 \end{pmatrix}$$

which has a negative discriminant, so $(1, 0)$ is a saddle. On the other hand, at $(2, 0)$, it is

$$\begin{pmatrix} -6 & 0 \\ 0 & -2 \end{pmatrix}$$

which has positive discriminant and a negative first entry, so its a local maximum. b) Use $ClassifyCriticalPoints[$ for $4x + 3y + x^3 + y^3 - x^4y - x^2y^2 + xy$. There are two saddle points. $(x, y) = (-1.631, -0.753)$, and $(x, y) = (1.356, 0.844)$.



Graph theorists look at the **Tutte polynomial** $f(x, y)$ of a network. We work with the Tutte polynomial

5 $$f(x, y) = x + 2x^2 + x^3 + y + 2xy + y^2$$

of the **Kite network**. Classify the two critical points using the second derivative test.

Remark. The polynomial is useful: $xf(1 - x, 0)$ tells in how many ways one can color the nodes of the network with x colors and $f(1, 1)$ tells how many spanning trees there are. This picture illustrates that the number of spanning trees of the kite graph is $f(1, 1) = 8$ as you see the 8 possible trees.

Solution:
The gradient is $\nabla f(x, y) = [1 + 4x + 3x^2 + 2y, 1 + 2x + 2y]$. The two given points are critical points. The point $(-2/3, 1/6)$ is a saddle point, the point $(0, -1/2)$ is a minimum. The discriminant at the first point is -4 at the second 4.

Main definitions

Standard assumption is that functions are smooth in the sense that all first and second partial derivatives are continuous.

A point (x_0, y_0) is a **critical point** of f if $\nabla f(x_0, y_0) = [0, 0]$.

Fermat's theorem: if f has a local maximum or local minimum at (x_0, y_0) then (x_0, y_0) is a critical point

Second derivative test: Assume (x_0, y_0) is a critical point. Define the discriminant $D = f_{xx}f_{yy} - f_{xy}^2$. If $D < 0$ then it is a saddle point. If $D > 0$, $f_{xx} < 0$ then (x_0, y_0) is a local maximum. If $D > 0$, $f_{xx} > 0$ then (x_0, y_0) is a local minimum.