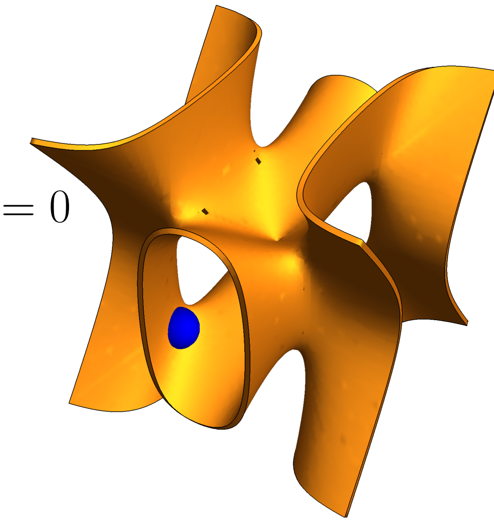
Homework 15: Tangent lines and planes

This homework is due Friday, 10/18.



1 a) Find the tangent plane to the surface

$$x^2 + y^2 - x^2y^2 - 2z^2 + x^2z^2 = 0$$

at the point $(x, y, z) = (-1, -2, -1)$.
b) Find the tangent line to the curve

$$x^2 + y^2 - x^2y^2 = -23$$

at the point $(x, y) = (3, 2)$.

2 In each of the following four conceptional problems, please answer briefly.
a) The figure 8 curve $f(x, y) = x^4 - x^2 + y^2 = 0$ has no tangent line at $(0, 0)$. Why? Isn't it perfectly smooth function $f(x, y)$?
b) The following statement is nonsense: "the tangent plane to the graph of $f(x, y, z) = x^2 + y^2 + z^2$ at $(x, y, z) = (1, 2, 3)$ is $2x + 4y + 6z = 28$". Modify it to make it a true statement.
c) Let $\vec{r}(u, v) = [u, v, g(u, v)]$ and $[a, b, c] = \vec{r}_u(1, 1) \times \vec{r}_v(1, 1)$. Let $f(x, y, z) = z - g(x, y)$ and $[A, B, C] = \nabla f(1, 1, g(1, 1))$. What is the relation between $[a, b, c]$ and $[A, B, C]$?
d) Given a closed curve $x^4 + y^8 = 2$, there is a point (x_0, y_0) where the gradient is parallel to $[3/5, 4/5]$. Are there two points?

3 a) Find an equation of the tangent plane to the parametric surface

$$\vec{r}(u, v) = [u^2, v^2, uv]$$

at the point $(u, v) = (1, 1)$.
b) The surface satisfies the equation $xy - z^2 = 0$. Find the tangent plane to this surface at the same point $(x, y, z) = (1, 1, 1)$ by computing the gradient.

4 Find an equation of the tangent plane and the normal line to the surface $x - z - 4\arctan(yz) = 0$ through the point $(1+\pi, 1, 1)$.

5 a) Show that the ellipsoid $6x^2 + 4y^2 + 2z^2 = 18$ and the sphere $x^2 + y^2 + z^2 - 8x - 6y - 8z + 24 = 0$ are tangent to each other at the point $(1, 1, 2)$, meaning that they have the same tangent plane at that point.
b) Find a surface different from a plane for which $x + y + 2z = 4$ is the tangent plane at the point $(1, 1, 1)$.

Main definitions

The **gradient** $\nabla f(x, y) = [f_x(x, y), f_y(x, y)]$ or $\nabla f(x, y, z) = [f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)]$ is the higher dimensional version of the first derivative $f'(x)$.

Theorem: Gradients are orthogonal to level sets.

The tangent line through (x_0, y_0) to a level curve $f(x, y) = c$ is $ax + by = d$, where $\nabla f(x_0, y_0) = [a, b]$ and d is obtained by plugging in the point. The tangent plane through (x_0, y_0, z_0) to a level surface $f(x, y, z) = C$ is $ax + by + cz = d$, where $\nabla f(x_0, y_0, z_0) = [a, b, c]$ and d is obtained by plugging in the point.

For parametrized surfaces $\vec{r}(u, v)$, the tangent plane is computed using the vectors $\vec{r}_u, \vec{r}_v$ are velocity vectors of grid curves and so tangent to the surface. The normal is $\vec{n} = \vec{r}_u \times \vec{r}_v = [a, b, c]$ and then get $ax + by + cz = d$, where d is obtained by plugging in the point $\vec{r}(u_0, v_0)$.