

Homework 14: Chain Rule

This homework is due Wednesday, 10/17/2019. There is no class on Monday, 10/15/2019.

- 1 a) Use the chain rule to find the derivative $\frac{d}{dt}f(\vec{r}(t))$ at $t = 1$ for

$$f(x, y) = \sin(x^7 + y^3) ,$$

where $\vec{r}(t) = [x(t), y(t)] = [t^5, 1/t]$. To do so, compute $\vec{r}'(1)$ and $\nabla f(\vec{r}(1))$ and then the dot product.

b) Now compute the derivative directly by differentiating $f(\vec{r}(t))$ directly with respect to t . You should get the same thing.

- 2 Find $dy/dx = y'(x)$ if x, y are related by

$$\sin(x) + \cos(y) = \sin(x) \cos(y) .$$

- 3 Find z_x and z_y for $yz = \log(x + z)$, where $\log = \ln$ is the natural log.

- 4 $\frac{d}{dt}f(\vec{r}(t)) = 25$ at $t = 0$ if $\vec{r}(t) = [t, t]$ and $\frac{d}{dt}f(\vec{r}(t)) = 11$ at $t = 0$. $\vec{r}(t) = [t, -t]$. Find the gradient of f at $(0, 0)$.

- 5 The Voltage V in a simple electrical circuit is slowly decreasing as the battery wears out. The resistance R is slowly increase as the resistor heats up. Use **Ohm's Law**, $V = IR$, to find how the current I is changing at the moment when $R = 400$, $I = 0.08$ $dV/dt = -0.01$ and $dR/dt = 0.03$.

Main definitions

Define the **gradient** $\nabla f(x, y) = [f_x(x, y), f_y(x, y)]$ or $\nabla f(x, y, z) = [f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)]$.

The **multivariable chain rule** is

$$\frac{d}{dt}f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t) .$$

When written out in two dimensions, this is

$$\frac{d}{dt}f(x(t), y(t)) = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t)$$

Example: a bug walks on $\vec{r}(t) = [\cos(t), \sin(t)]$ on a plane with temperature $f(x, y) = x^2 + 5y$. Find the temperature change $d/dt f(\vec{r}(t))$ at $(1, 0)$. **Solution:** either compose $f(\vec{r}(t)) = \cos^2(t) + 5 \sin(t)$ and differentiate at $t = 0$ to get $d/dt f(\vec{r}(t)) = 5 \cos(0) = 5$. Or then find $\vec{r}'(0) = [0, 1]$ and the gradient $\nabla f(x, y) = [2x, 5]$ which is $[0, 5]$ at $(1, 0)$. The chain rule assures that the dot product is the same.

We can use the chain rule for implicit differentiation:

Implicit differentiation: If $f(x, y) = c$ is a curve and near some point (x_0, y_0) $y = y(x)$, we can compute $y' = -f_x/f_y$.

In three dimensions, the **implicit differentiation formulas** derived from the chain rule are:

$$z_x(x, y) = -f_x(x, y, z)/f_z(x, y, z)$$

$$z_y(x, y) = -f_y(x, y, z)/f_z(x, y, z)$$