Homework 14: Chain Rule

This homework is due Wednesday, 10/17/2019. There is no class on Monday, 10/15/2019.

1 a) Use the chain rule to find the derivative $\frac{d}{dt}f(\vec{r}(t))$ at $t = 1$ for

$$f(x, y) = \sin(x^7 + y^3) ,$$

where $\vec{r}(t) = [x(t), y(t)] = [t^5, 1/t]$. To do so, compute $\vec{r}\,'(1)$ and $\nabla f(\vec{r}(1))$ and then the dot product.
b) Now compute the derivative directly by differentiating $f(\vec{r}(t))$ directly with respect to t. You should get the same thing.

Solution:
a) $\vec{r}\,'(t) = [5t^4, -1/t^2]$ so that $\vec{r}(0) = [5, -1]$. $f_x = \cos(x^7 + y^3)7x^6$, $f_y = \cos(x^7+y^3)3y^2$. At $(1, 1)$ this is $[7\cos(2), 3\cos(2)]$. The result is $32\cos(2)$. b) $f(\vec{r}(t)) = \sin(t^{35}+1/t^3)$. The derivative is $32\cos(2)$.

2 Find $dy/dx = y'(x)$ if x, y are related by

$$\sin(x) + \cos(y) = \sin(x)\cos(y) .$$

Solution:
Implicit differentiation with respect to x yields:

$$\cos(x) - \sin(y)y' = \cos(x)\cos(y) - \sin(x)\sin(y)y'$$

Therefore,

$$y' = \frac{\cos(x)\cos(y) - \cos(x)}{\sin(x)\sin(y) - \sin(y)}.$$

3 Find z_x and z_y for $yz = \log(x+z)$, where $\log = \ln$ is the natural log.

Solution:
Implicit differentiation with respect to x gives us:

$$yz_x = \frac{1+z_x}{x+z}$$

Hence $yz_x(x+z) = 1 + z_x$ or

$$z_x = \frac{1}{y(x+z)-1}.$$

Implicit differentiation with respect to y gives us:

$$z + yz_y = \frac{z_y}{x+z}$$

Hence $(x+z)z + (x+z)yz_y = z_y$ or

$$z_y = \frac{(x+z)z}{1-(x+z)y}.$$

4 $\frac{d}{dt}f(\vec{r}(t)) = 25$ at $t = 0$ if $\vec{r}(t) = [t, t]$ and $\frac{d}{dt}f(\vec{r}(t)) = 11$ at $t = 0$. $\vec{r}(t) = [t, -t]$. Find the gradient of f at $(0, 0)$.

Solution:
Let $a = f_x$ and $b = f_y$. Then the assumptions give $a + b = 25$ and $a - b = 11$. Therefore $a = 18, b = 7$.

5 The Voltage V in a simple electrical circuit is slowly decreasing as the battery wears out. The resistance R is slowly increase as the resistor heats up. Use **Ohm's Law**, $V = IR$, to find how the current I is changing at the moment when $R = 400$, $I = 0.08$

$dV/dt = -0.01$ and $dR/dt = 0.03$.

Solution:
By the chain rule,

$$-0.01 = \frac{\partial V}{\partial t} = \frac{\partial I}{\partial t}R + I\frac{\partial R}{\partial t} = 400\frac{\partial I}{\partial t} + 0.08 \cdot 0.03.$$

Thus, $\frac{\partial I}{\partial t} \approx -0.000031$.

Main definitions

Define the **gradient** $\nabla f(x,y) = [f_x(x,y), f_y(x,y)]$ or $\nabla f(x,y,z) = [f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)]$.

The **multivariable chain rule** is

$$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t) \ .$$

When written out in two dimensions, this is

$$\frac{d}{dt} f(x(t), y(t)) = f_x(x(t), y(t)) x'(t) + f_y(x(t), y(t)) y'(t)$$

Example: a bug walks on $\vec{r}(t) = [\cos(t), \sin(t)]$ on a plane with temperature $f(x,y) = x^2 + 5y$. Find the temperature change $d/dt f(\vec{r}(t))$ at $(1,0)$. **Solution:** either compose $f(\vec{r}(t)) = \cos^2(t) + 5\sin(t)$ and differentiate at $t = 0$ to get $d/dt f(\vec{r}(t)) = 5\cos(0) = 5$. Or then find $\vec{r}\,'(0) = [0,1]$ and the gradient $\nabla f(x,y) = [2x, 5]$ which is $[0,5]$ at $(1,0)$. The chain rule assures that the dot product is the same.
We can use the chain rule for implicit differentiation:

Implicit differentiation: If $f(x,y) = c$ is a curve and near some point (x_0, y_0) $y = y(x)$, we can compute $y' = -f_x/f_y$.

In three dimensions, the **implicit differentiation formulas** derived from the chain rule are:

$$z_x(x,y) = -f_x(x,y,z)/f_z(x,y,z)$$

$$z_y(x,y) = -f_y(x,y,z)/f_z(x,y,z)$$