Homework 12: Partial differential equations

This homework is due Wednesday, 10/9/2019.

1 a) The following functions solve either the **Laplace equation** $u_{xx} + u_{yy} = 0$ or the **wave equation** $u_{xx} - u_{yy} = 0$. Decide in each case. Possible answers are "none", "both", "Wave equation" or "Laplace equation". As usual $\log = \ln$ is the natural log.

a) $u = 5x^2 - 5y^2$

b) $u = \sin(x - y)$

c) $u = 7 \log \sqrt{x^2 + y^2}$

d) $u = e^{-x} \cos y - e^{-y} \cos x$

e) $u = \sin(5x) \sin(5y)$

f) $u = \left(\frac{y}{y^2 - x^2}\right)$

g) $u = x^4 - 6x^2y^2 + y^4$

h) $u = x^3 - 3xy^2$

Solution:

(a) Laplace equation.

(b) Wave equation

(c) Laplace equation. Hint: Write this as $(7/2)\log(x^2 + y^2)$.

(d) Laplace's Equation.

(e) Wave equation

(f) Wave equation.

(g) Laplace equation.

(h) Laplace equation

2 The differential equation

$$f_t = f - xf_x - x^2 f_{xx}$$

is an example of the **infamous Black-Scholes equation**. Here $f(x, t)$ is the price of a call option and x the stock price and t is time.
a) Verify that $f(t, x) = x$, $f(t, x) = (1+x^2)/(2x)$ and $f(t, x) = e^t$ solve this PDE. b) Verify that $e^t \log(x)$ solves Black-Scholes.
c) Verify that $e^{-3t}x^2$ solves Black Scholes.

Solution:
a) Just differentiate in each case.
b) dito

3 a) Show that the **Cobb-Douglas** production function $P = L^\alpha K^\beta$ satisfies the equation

$$L\frac{\partial P}{\partial L} + K\frac{\partial P}{\partial K} = (\alpha+\beta)P \; .$$

The constants α and β are fixed. L is labor and K is capital.
b) Verify that $f(x,y) = \sqrt{x^2+y^2}$ satisfies the **Eikonal equation** $f_x^2 + f_y^2 = 1$. This partial differential equation is important in optics. We will later see that it can be written as $|\nabla f| = 1$.

Solution:
$P = L^\alpha K^\beta$, so $\dfrac{\partial P}{\partial L} = \alpha L^{\alpha-1}K^\beta$ and $\dfrac{\partial P}{\partial K} = \beta L^\alpha K^{\beta-1}$. Then $L\dfrac{\partial P}{\partial L} + K\dfrac{\partial P}{\partial K} = L(\alpha L^{\alpha-1}K^\beta) + K(\beta L^\alpha K^{\beta-1}) = \alpha L^{1+\alpha-1}K^\beta + \beta L^\alpha K^{1+\beta-1} = (\alpha+\beta)L^\alpha K^\beta = (\alpha+\beta)P$.

4 Verify that for any real constant b, the function $f(x,t) = e^{-bt}\cos(x+t)$ satisfies the driven transport equation $f_t(x,t) = f_x(x,t) - bf(x,t)$. This PDE is sometimes called the **advection equation** with damping b.

5 The partial differential equation

$$f_t + ff_x = f_{xx}$$

called **Burgers equation** (with damping) describes waves at the beach. In higher dimensions, it leads to the **Navier-Stokes** equation which is used to describe the weather. Use Mathematica to verify that the function

$$f(t,x) = \frac{\left(\frac{1}{t}\right)^{3/2} x e^{-\frac{x^2}{4t}}}{\sqrt{\frac{1}{t}} e^{-\frac{x^2}{4t}} + 1}$$

solves the Burgers equation.

Solution:
Verify the Burgers equation using Mathematica!

An equation for an unknown function $f(x,y)$ which involves partial derivatives with respect to at least two different variables is called a **partial differential equation**. You have to be able to verify whether some function solves a partial differential equation. You will also have to know some basic differential equations: the Laplace, the heat, the wave and the transport and the Burger equation. In lecture we provide some background on these equations and meaning which helps to know them without blindly memorizing them.