Homework 10: Functions

This homework is due Friday, 10/5/2019. The topic of function reviews also a bit of vector-valued functions like curves $\vec{r}(t)$ and parametrized surfaces $\vec{r}(u,v) = [x(u,v), y(u,v), z(u,v)]$.

1 For which functions is the domain of f the entire plane?
a) $f(x,y) = ye^{1/x}$
b) $f(x,y) = \log(1 + |x+y|)$
c) $f(x,y) = \sqrt{(|x-y|)}$
d) $f(x,y) = e^x/(1 + \sin(y))$

2 When defining parametrized surfaces $\vec{r}(u,v) = [x(u,v), y(u,v), z(u,v)]$ we use functions of two variables in all coordinates. Coming up with good parametrizations can be a bit tricky sometimes. We look at the surface S given as a level surface $|x|^{2/5} + |y|^{2/5} + |z|^{2/5} = 1$. Our goal is to parametrize S. To do so, start with the case when $2/5$ is replaced by 2. Then modify the parametrization by changing each function in a suitable way. All functions $x(u,v)$, $y(u,v)$ and $z(u,v)$ will be nice and differentiable (we talk about derivatives in the next hour).

3 If $\vec{r}(t)$ is a parametrization, the resulting "curve" is defined as all the values of $\vec{r}(t)$, when t is a real number $t \in \mathbb{R}$. Look at the range of $\vec{r}$ in the following two cases?
a) $\vec{r}(t) = [\cos(t), \sin(2t)]$?
b) $\vec{r}(t) = [\cos(t), \sin(\sqrt{2}t)]$
The "curves" in the two examples look different. Why? To investigate, you might want to plot the two curves with a computer for t in an interval like $[0, 200\pi]$.

4 Consider the function $f(x,y) = x/\sqrt{x^2+y^2}$. It has the property that the two level curves $f = 0$ and $f = 1$ intersect at $(0,0)$.
a) Draw $f = 0$ and parametrize it as $\vec{r}_1(t)$ so that $\vec{r}_1(0) = [0,0]$.
b) Draw $f = 1$ and parametrize it as $\vec{r}_2(t)$ so that $\vec{r}_2(0) = [0,0]$.

c) Is there a value $a = f(0,0)$ so that $t \to f(\vec{r}_1(t))$ and $t \to f(\vec{r}_2(t))$ are both continuous functions in t with the requirement $f(\vec{r}_1(0)) = f(\vec{r}_2(t) = a$? If yes, what is the value a? If no, why is there no value?

5 If we want to investigate whether the limiting value of $f(x,y)$ at $(0,0)$ is defined, one strategy is to use polar coordinates and check whether $\lim_{r\to 0} f(r\cos(\theta), r\sin(\theta))$ is defined. In the previous problem you have seen that there was no way to define $f(0,0)$ to make f continuous. The problem can be that approaching $(0,0)$ along curves from different directions gives different values. Decide in the following three cases, a limiting value $\lim_{(x,y)\to(0,0)} f(x,y)$ exists. You might want to recall the l'Hôpital rule.

a) $f(x,y) = \frac{\cos(x^2+y^2)-1}{x^2+y^2}$ if $(x,y) \neq (0,0)$ and $f(x,y) = 1/2$ if $(x,y) = 0$.

b) $f(x,y) = \frac{x^6-y^6}{(x^2+y^2)^2}$ if $(x,y) \neq (0,0)$ and $f(x,y) = 0$ if $(x,y) = 0$.

c) $f(x,y) = \frac{x^2-y^2}{(x^2+y^2)}$ if $(x,y) \neq (0,0)$ and $f(x,y) = 0$ if $(x,y) = 0$.

Main points

The **domain** of a function or parametrization is the place where the function is defined. Notions of continuity for functions are similar in the multi-variable case: for example, a function $f(x,y)$ with domain R is **continuous at** $(a,b) \in R$ if $f(x,y) \to f(a,b)$ for all choices $(x,y) \to (a,b)$. To decide about continuity of $f(x,y)$, polar coordinates $x = r\cos(\theta), y = r\sin(\theta)$ can help. For $f(x,y) = \frac{x^2y^2}{x^2+y^2}$ for example, we have $f(r,\theta) = \cos^2(\theta) - \sin^2(\theta)$. As arbitrarily close to $(0,0)$, the function takes any value in $[-1,1]$, the function is not continuous. For vector valued functions like parametrizations of surfaces $\vec{r}(u,v) = [x(u,v), y(u,v), z(u,v)]$, the domain is the intersection of the domains for the three functions.